PROPOSTAS DE RESOLUÇÃO

do Caderno de Exercícios

1. Trigonometria

1. Resolução de problemas que envolvem triângulos

Exercícios – páginas 3 a 7

1.

1.1. $\text{sen } 45^\circ \times \cos 45^\circ - \cos 60^\circ - \text{sen } 60^\circ =$

$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} - \frac{\sqrt{3}}{2} =$

$= \frac{1}{2} - \frac{1}{2} - \frac{\sqrt{3}}{2} =$

$= -\frac{\sqrt{3}}{2}$

1.2. $\text{sen } 30^\circ + \text{tg } 60^\circ \times \cos 30^\circ - \text{tg } 45^\circ \times \cos 60^\circ =$

$= \frac{1}{2} + \sqrt{3} \times \frac{\sqrt{3}}{2} - 1 \times \frac{1}{2} =$

$= \frac{1}{2} + \frac{3}{2} - \frac{1}{2} =$

$= \frac{3}{2}$

1.3. $\text{tg } 30^\circ - \text{tg } 60^\circ + \cos 30^\circ + \text{sen } 60^\circ =$

$= \frac{\sqrt{3}}{3} - \sqrt{3} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} =$

$= \frac{\sqrt{3}}{3} - \sqrt{3} + \sqrt{3} =$

$= \frac{\sqrt{3}}{3}$

1.4. $\text{tg } 45^\circ + \cos 30^\circ - \text{tg } 60^\circ \times \text{sen } 60^\circ =$

$= 1 + \frac{\sqrt{3}}{2} - \sqrt{3} \times \frac{\sqrt{3}}{2} =$

$= 1 + \frac{\sqrt{3}}{2} - \frac{3}{2} =$

$= \frac{\sqrt{3}}{2} - \frac{1}{2} =$

$= \frac{\sqrt{3} - 1}{2}$

2.

2.1. $\text{sen } 60^\circ = \frac{a}{6} \Leftrightarrow a = 6 \times \text{sen } 60^\circ$

$\Leftrightarrow a = 6 \times \frac{\sqrt{3}}{2}$

$\Leftrightarrow a = 3\sqrt{3}$

$\cos 60^\circ = \frac{b}{6} \Leftrightarrow b = 6 \times \cos 60^\circ$

$\Leftrightarrow b = 6 \times \frac{1}{2}$

$\Leftrightarrow b = 3$

Logo, $a = 3\sqrt{3}$ e $b = 3$.

2.2. Como o triângulo é retângulo e um dos ângulos agudos tem amplitude 45°, então o outro ângulo agudo também tem amplitude 45° e o triângulo é isósceles. Logo, $a = 6$.

$\text{sen } 45^\circ = \frac{6}{b} \Leftrightarrow b = \frac{6}{\text{sen } 45^\circ}$

$\Leftrightarrow b = \frac{6}{\frac{\sqrt{2}}{2}}$

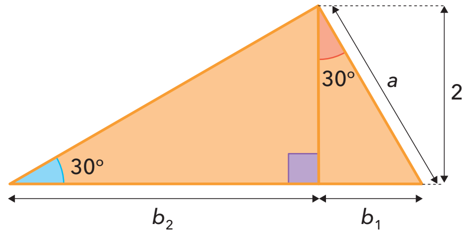
$\Leftrightarrow b = \frac{12}{\sqrt{2}}$

$\Leftrightarrow b = \frac{12\sqrt{2}}{2}$

$\Leftrightarrow b = 6\sqrt{2}$

Logo, $a = 6$ e $b = 6\sqrt{2}$.

2.3.



$\cos 30^\circ = \frac{2}{a} \Leftrightarrow a = \frac{2}{\cos 30^\circ}$

$\Leftrightarrow a = \frac{2}{\frac{\sqrt{3}}{2}}$

$\Leftrightarrow a = \frac{4}{\sqrt{3}}$

$\Leftrightarrow a = \frac{4\sqrt{3}}{2}$

$\text{tg } 30^\circ = \frac{b_1}{2} \Leftrightarrow b_1 = 2 \times \text{tg } 30^\circ$

$\Leftrightarrow b_1 = 2 \times \frac{\sqrt{3}}{3}$

$\Leftrightarrow b_1 = \frac{2\sqrt{3}}{3}$

$\text{tg } 30^\circ = \frac{2}{b_2} \Leftrightarrow b_2 = \frac{2}{\text{tg } 30^\circ}$

$\Leftrightarrow b_2 = \frac{2}{\frac{\sqrt{3}}{3}}$

$\Leftrightarrow b_2 = \frac{6\sqrt{3}}{3}$

$\Leftrightarrow b_2 = 2\sqrt{3}$

Assim, $b = b_1 + b_2 = \frac{2\sqrt{3}}{3} + 2\sqrt{3} = \frac{8\sqrt{3}}{3}$.

Logo, $a = \frac{4\sqrt{3}}{3}$ e $b = \frac{8\sqrt{3}}{3}$.

2.4. $\cos 45^\circ = \frac{a}{\sqrt{2}} \Leftrightarrow a = \sqrt{2} \times \cos 45^\circ$

$\Leftrightarrow a = \sqrt{2} \times \frac{\sqrt{2}}{2}$

$\Leftrightarrow a = 1$

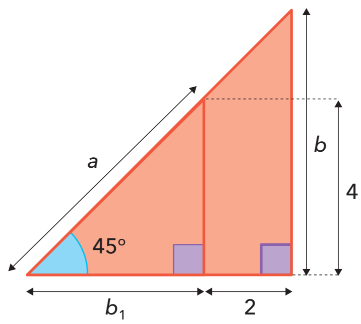
$\cos 60^\circ = \frac{a}{b} \Leftrightarrow b = \frac{a}{\cos 60^\circ}$

$\Leftrightarrow b = \frac{1}{\frac{1}{2}}$

$\Leftrightarrow b = 2$

Logo, $a = 1$ e $b = 2$.

2.5.



$\text{sen } 45^\circ = \frac{4}{a} \Leftrightarrow a = \frac{4}{\text{sen } 45^\circ}$

$\Leftrightarrow a = \frac{4}{\frac{\sqrt{2}}{2}}$

$\Leftrightarrow a = \frac{8}{\sqrt{2}}$

$\Leftrightarrow a = \frac{8\sqrt{2}}{2}$

$\Leftrightarrow a = 4\sqrt{2}$

Como o triângulo é retângulo e um dos ângulos agudos tem amplitude 45º, então o outro ângulo agudo também tem amplitude 45º e o triângulo é isósceles. Logo, $b_1 = 4$.
Pelo mesmo motivo, $b = b_1 + 2 = 4 + 2 = 6$.
Logo, $a = 4\sqrt{2}$ e $b = 6$.

2.6. $\cos 60^\circ = \frac{4}{a} \Leftrightarrow a = \frac{4}{\cos 60^\circ} \Leftrightarrow a = \frac{4}{\frac{1}{2}} \Leftrightarrow a = 8$

O triângulo à esquerda tem um ângulo de amplitude 120º e dois ângulos de amplitude 30º, logo é um triângulo isósceles e, por isso, $b = a$, ou seja, $b = 8$.
Logo, $a = 8$ e $b = 8$.

3.
3.1. $\text{tg } 30^\circ = \frac{\overline{AD}}{\overline{AE}} \Leftrightarrow \frac{\sqrt{3}}{3} = \frac{\frac{7\sqrt{3}}{3}}{\overline{AE}} \Leftrightarrow \overline{AE} = \frac{\frac{7\sqrt{3}}{3}}{\frac{\sqrt{3}}{3}} \Leftrightarrow \overline{AE} = 7$

$\cos 60^\circ = \frac{\overline{BE}}{\overline{CE}} \Leftrightarrow \frac{1}{2} = \frac{\overline{BE}}{8} \Leftrightarrow \overline{BE} = 8 \times \frac{1}{2} \Leftrightarrow \overline{BE} = 4$

Assim, $\overline{AB} = \overline{AE} + \overline{BE} = 7 + 4 = 11$.

$\text{sen } 60^\circ = \frac{\overline{BC}}{\overline{CE}} \Leftrightarrow \frac{\sqrt{3}}{2} = \frac{\overline{BC}}{8} \Leftrightarrow \overline{BC} = 8 \times \frac{\sqrt{3}}{2}$

$\Leftrightarrow \overline{BC} = 4\sqrt{3}$

Logo, $\overline{AB} = 11$ u.c. e $\overline{BC} = 4\sqrt{3}$ u.c.

3.2. $\text{sen } 30^\circ = \frac{\overline{AD}}{\overline{DE}} \Leftrightarrow \frac{1}{2} = \frac{\frac{7\sqrt{3}}{3}}{\overline{DE}} \Leftrightarrow \overline{DE} = \frac{\frac{7\sqrt{3}}{3}}{\frac{1}{2}}$

$\Leftrightarrow \overline{DE} = \frac{14\sqrt{3}}{3}$

Pelo teorema de Pitágoras, $\overline{CD}^2 = \overline{DE}^2 + \overline{CE}^2$.

Assim, $\overline{CD}^2 = \left(\frac{14\sqrt{3}}{3}\right)^2 + 8^2 \Leftrightarrow \overline{CD}^2 = \frac{196}{3} + 64$

$\Leftrightarrow \overline{CD}^2 = \frac{388}{3}$

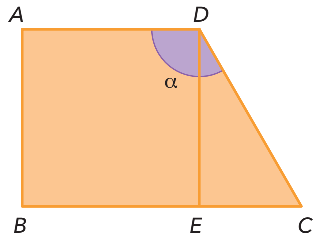
Logo, $\overline{CD} = \sqrt{\frac{388}{3}} = \frac{2\sqrt{291}}{3}$.

Assim, $P_{[ABCD]} = \overline{AD} + \overline{AB} + \overline{BC} + \overline{CD} =$

$= \frac{7\sqrt{3}}{3} + 11 + 4\sqrt{3} + \frac{2\sqrt{291}}{3} =$

$= \left(11 + \frac{19\sqrt{3}}{3} + \frac{2\sqrt{291}}{3}\right)$ u.c.

4.



$\overline{DE} = \overline{AB} = 6$

$C\hat{D}E = 120^\circ - 90^\circ = 30^\circ$

$\text{tg } 30^\circ = \frac{\overline{CE}}{\overline{DE}} \Leftrightarrow \frac{\sqrt{3}}{3} = \frac{\overline{CE}}{6} \Leftrightarrow \overline{CE} = 6 \times \frac{\sqrt{3}}{3}$

$\Leftrightarrow \overline{CE} = 2\sqrt{3}$

$\overline{BC} = \overline{BE} + \overline{CE} = 6 + 2\sqrt{3}$

$\cos 30^\circ = \frac{\overline{DE}}{\overline{CD}} \Leftrightarrow \frac{\sqrt{3}}{2} = \frac{6}{\overline{CD}} \Leftrightarrow \overline{CD} = \frac{6}{\frac{\sqrt{3}}{2}}$

$\Leftrightarrow \overline{CD} = \frac{12}{\sqrt{3}} \Leftrightarrow \overline{CD} = 4\sqrt{3}$

Assim,
$P_{[ABCD]} = \overline{AB} + \overline{BC} + \overline{CD} + \overline{AD} =$

$= 6 + 6 + 2\sqrt{3} + 4\sqrt{3} + 6 =$

$= (18 + 6\sqrt{3})$ u.c.

5. $\text{tg } 30^\circ = \frac{\overline{CE}}{\overline{CD}} \Leftrightarrow \frac{\sqrt{3}}{3} = \frac{\overline{CE}}{3\sqrt{3}}$

$\Leftrightarrow \overline{CE} = \frac{\sqrt{3}}{3} \times 3\sqrt{3}$

$\Leftrightarrow \overline{CE} = 3$

$\overline{BC} = \overline{CE} = 3$

$\overline{BD} = \overline{BD} + \overline{CD} = 3 + 3\sqrt{3}$

$\text{tg } 30^\circ = \frac{\overline{AB}}{\overline{BD}} \Leftrightarrow \frac{\sqrt{3}}{3} = \frac{\overline{AB}}{3 + 3\sqrt{3}}$

$\Leftrightarrow \overline{AB} = \frac{\sqrt{3}}{3} \times (3 + 3\sqrt{3})$

$\Leftrightarrow \overline{AB} = \sqrt{3} + 3$

Assim,

$A_{[ABCE]} = \frac{\overline{AB} + \overline{CE}}{2} \times \overline{BC} =$

$= \frac{\sqrt{3} + 3 + 3}{2} \times 3 =$

$= \frac{3\sqrt{3} + 18}{2} =$

$= \left(9 + \frac{3\sqrt{3}}{2}\right)$ u.a.

6. $\cos \alpha = \frac{\overline{AD}}{\overline{AE}} \Leftrightarrow \frac{4}{5} = \frac{12}{\overline{AE}} \Leftrightarrow \overline{AE} = \frac{12}{\frac{4}{5}} \Leftrightarrow \overline{AE} = 15$

Pelo teorema de Pitágoras, $\overline{AE}^2 = \overline{AD}^2 + \overline{DE}^2$.

Assim, $15^2 = 12^2 + \overline{DE}^2 \Leftrightarrow \overline{DE}^2 = 225 - 144$

$\Leftrightarrow \overline{DE}^2 = 81$

Logo, $\overline{DE} = 9$.
$\overline{EC} = \overline{DC} - \overline{DE} = 12 - 9 = 3$
Assim,
$P_{[ABCE]} = \overline{AB} + \overline{BC} + \overline{CE} + \overline{AE} =$
$= 12 + 12 + 3 + 15 = 42$ cm

7. O triângulo [ABC] está inscrito numa semicircunferência, logo é um triângulo retângulo.

$\cos\alpha = \frac{\overline{BC}}{\overline{AC}} \Leftrightarrow \cos\alpha = \frac{\overline{BC}}{2r} \Leftrightarrow \overline{BC} = 2r\cos\alpha$

$\text{sen}\,\alpha = \frac{\overline{AB}}{\overline{AC}} \Leftrightarrow \text{sen}\,\alpha = \frac{\overline{AB}}{2r} \Leftrightarrow \overline{AB} = 2r\,\text{sen}\,\alpha$

$A_{[ABC]} = \frac{\overline{AB} + \overline{BC}}{2} = \frac{2r\,\text{sen}\,\alpha \times 2r\cos\alpha}{2} =$
$= 2r^2\,\text{sen}\,\alpha\cos\alpha$

Assim, $A_{[ABCD]} = 2 \times 2r^2\,\text{sen}\,\alpha\cos\alpha = 4r^2\,\text{sen}\,\alpha\cos\alpha$.
Logo, a área da região representada a sombreado é dada por:
$\pi \times r^2 - 4r^2\,\text{sen}\,\alpha\cos\alpha = r^2(\pi - 4\,\text{sen}\,\alpha\cos\alpha)$

8. $\text{tg}\,64^\circ = \frac{\overline{AD}}{\overline{AB}} \Leftrightarrow \overline{AD} = \overline{AB}\,\text{tg}\,64^\circ$

$\text{tg}\,35^\circ = \frac{\overline{AD}}{68 + \overline{AB}} \Leftrightarrow (68 + \overline{AB})\,\text{tg}\,35^\circ = \overline{AD}$

$\Leftrightarrow 68\,\text{tg}\,35^\circ + \overline{AB}\,\text{tg}\,35^\circ = \overline{AB}\,\text{tg}\,64^\circ$
$\Leftrightarrow \overline{AB}\,\text{tg}\,64^\circ - \overline{AB}\,\text{tg}\,35^\circ = 68\,\text{tg}\,35^\circ$
$\Leftrightarrow \overline{AB}\,(\text{tg}\,64^\circ - \text{tg}\,35^\circ) = 68\,\text{tg}\,35^\circ$
$\Leftrightarrow \overline{AB} = \frac{68\,\text{tg}\,35^\circ}{\text{tg}\,64^\circ - \text{tg}\,35^\circ}$

Assim, $\overline{AD} = \overline{AB}\,\text{tg}\,64^\circ = \frac{68\,\text{tg}\,35^\circ}{\text{tg}\,64^\circ - \text{tg}\,35^\circ}\,\text{tg}\,64^\circ \approx 72$.

Logo, a altura do edifício é 72 metros.

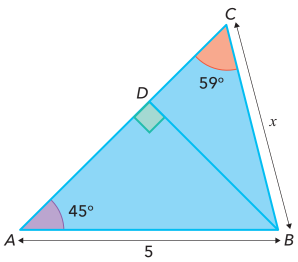
9. $\text{tg}\,59^\circ = \frac{\overline{CD}}{\overline{AD}} \Leftrightarrow \overline{CD} = \overline{AD}\,\text{tg}\,59^\circ$

$\text{tg}\,44^\circ = \frac{\overline{CD}}{\overline{AB}} \Leftrightarrow \overline{CD} = \overline{AB}\,\text{tg}\,44^\circ$
$\Leftrightarrow \overline{AD}\,\text{tg}\,59^\circ = (76 - \overline{AD})\,\text{tg}\,44^\circ$
$\Leftrightarrow \overline{AD}\,\text{tg}\,59^\circ = 76\,\text{tg}\,44^\circ - \overline{AD}\,\text{tg}\,44^\circ$
$\Leftrightarrow \overline{AD}\,\text{tg}\,59^\circ + \overline{AD}\,\text{tg}\,44^\circ = 76\,\text{tg}\,44^\circ$
$\Leftrightarrow \overline{AD}\,(\text{tg}\,59^\circ + \text{tg}\,44^\circ) = 76\,\text{tg}\,44^\circ$
$\Leftrightarrow \overline{AD} = \frac{76\,\text{tg}\,44^\circ}{\text{tg}\,59^\circ + \text{tg}\,44^\circ}$

Assim, $\overline{CD} = \frac{76\,\text{tg}\,44^\circ}{\text{tg}\,59^\circ + \text{tg}\,44^\circ} \times \tan 59^\circ \approx 46$.

Logo, a altura do farol é 46 metros.

10.
10.1.

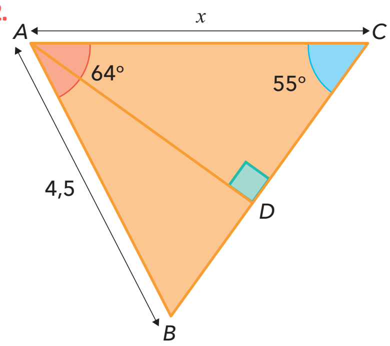


$\text{sen}\,45^\circ = \frac{\overline{BD}}{5} \Leftrightarrow \overline{BD} = 5\,\text{sen}\,45^\circ \Leftrightarrow \overline{BD} = \frac{5\sqrt{2}}{2}$

$\text{sen}\,59^\circ = \frac{\overline{BD}}{x} \Leftrightarrow x = \frac{\overline{BD}}{\text{sen}\,59^\circ} \Leftrightarrow x = \frac{5\sqrt{2}}{2\,\text{sen}\,59^\circ}$

Logo, $x \approx 4{,}1$.

10.2.



$D\hat{A}C = 180^\circ - 64^\circ - 55^\circ = 61^\circ$

$\text{sen}\,61^\circ = \frac{\overline{AD}}{4{,}5} \Leftrightarrow \overline{AD} = 4{,}5\,\text{sen}\,61^\circ$

$\text{sen}\,55^\circ = \frac{\overline{AD}}{x} \Leftrightarrow x = \frac{4{,}5\,\text{sen}\,61^\circ}{\text{sen}\,55^\circ}$

Logo, $x \approx 4{,}8$.

11.
11.1. Seja h o comprimento da altura do triângulo [ABC] relativamente ao vértice C.
Tem-se que:

$\text{sen}\,60^\circ = \frac{h}{b} \Leftrightarrow \frac{\sqrt{3}}{2} = \frac{h}{b} \Leftrightarrow h = \frac{\sqrt{3}}{2}b$

Tem-se ainda que:

$\text{sen}\,45^\circ = \frac{h}{a} \Leftrightarrow \frac{\sqrt{2}}{2} = \frac{h}{a} \Leftrightarrow h = \frac{\sqrt{2}}{2}a$

Assim, $\frac{\sqrt{3}}{2}b = \frac{\sqrt{2}}{2}a$.

Como $a - b = \sqrt{3} \Leftrightarrow b = a - \sqrt{3}$, então:

$\frac{\sqrt{3}}{2}(a - \sqrt{3}) = \frac{\sqrt{2}}{2}a \Leftrightarrow \frac{\sqrt{3}}{2}a - \frac{3}{2} = \frac{\sqrt{2}}{2}a$
$\Leftrightarrow \sqrt{3}a - \sqrt{2}a = 3$
$\Leftrightarrow a(\sqrt{3} - \sqrt{2}) = 3$
$\Leftrightarrow a = \frac{3}{\sqrt{3} - \sqrt{2}}$
$\Leftrightarrow a = \frac{3(\sqrt{3} + \sqrt{2})}{3 - 2}$
$\Leftrightarrow a = 3\sqrt{3} + 3\sqrt{2}$

12.
12.1. $\cos(A\hat{B}V) = \frac{\frac{1}{2} \times \frac{2}{3}\overline{AV}}{\overline{AV}} = \frac{1}{3}$

Logo, $A\hat{B}V = \cos^{-1}\left(\frac{1}{3}\right) \approx 70{,}5^\circ$.

12.2. $A\hat{V}B = 180^\circ - 2 \times \cos^{-1}\left(\frac{1}{3}\right) \approx 38{,}9^\circ$

12.3. Pelo teorema de Pitágoras, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$.

Assim, $\overline{AC}^2 = \left(\frac{2}{3}\overline{AV}\right)^2 + \left(\frac{2}{3}\overline{AV}\right)^2$

$\Leftrightarrow \overline{AC}^2 = \frac{4}{9}\overline{AV}^2 + \frac{4}{9}\overline{AV}^2 \Leftrightarrow \overline{AC}^2 = \frac{8}{9}\overline{AV}^2$

Logo, $\overline{AC} = \frac{2\sqrt{2}}{3}\overline{AV}$.

$\text{sen}\left(\frac{A\hat{V}C}{2}\right) = \frac{\frac{1}{2} \times \frac{2\sqrt{2}}{3}\overline{AV}}{\overline{AV}} = \frac{\sqrt{2}}{3}$

Logo, $\frac{A\hat{V}C}{2} = \text{sen}^{-1}\left(\frac{\sqrt{2}}{3}\right)$

$\Leftrightarrow A\hat{V}C = 2 \times \text{sen}^{-1}\left(\frac{\sqrt{2}}{3}\right) \approx 56{,}3^\circ$.

12.4. $A\hat{C}V = \dfrac{180^\circ - 2\,\text{sen}^{-1}\left(\frac{\sqrt{2}}{3}\right)}{2} \approx 61{,}9^\circ$

13. Como a reta AC é a mediatriz de $[BD]$, então $\overline{BE} = \overline{DE}$ e $A\hat{B}E = A\hat{D}E = \alpha$. Além disso, $[AC]$ é um diâmetro da circunferência. Logo, $[ABC]$ e $[ADL]$ são triângulos retângulos.

$\text{sen}\,\alpha = \frac{\overline{AE}}{\overline{AB}} \Leftrightarrow \overline{AE} = 2\,\text{sen}\,\alpha$

$\cos\alpha = \frac{\overline{BE}}{\overline{AB}} \Leftrightarrow \overline{BE} = 2\cos\alpha$

Uma vez que $A\hat{B}E = A\hat{D}E = \alpha$, então a amplitude do arco AB é igual à amplitude do arco AC, que, por sua vez, corresponde ao ângulo inscrito BCA. Logo, $B\hat{C}A = \alpha$ e, analogamente, $D\hat{C}A = \alpha$.
Além disso, o triângulo $[ACB]$ e o triângulo $[EBA]$ são semelhantes, visto que ambos são triângulos retângulos e têm ambos um ângulo de amplitude α. Pelo mesmo motivo, os triângulos $[ACD]$ e $[ACE]$ são semelhantes.

Assim, $\frac{\overline{AE}}{\overline{AB}} = \frac{\overline{BE}}{\overline{BC}} \Leftrightarrow \frac{2\,\text{sen}\,\alpha}{2} = \frac{2\cos\alpha}{\overline{BC}}$

$\Leftrightarrow \overline{BC} = \frac{2 \times 2\cos\alpha}{2\,\text{sen}\,\alpha}$

$\Leftrightarrow \overline{BC} = \frac{2\cos\alpha}{\text{sen}\,\alpha}$.

Então, $A_{[ABCD]} = 2 \times A_{[ABC]} = 2 \times \frac{\overline{AB} \times \overline{BC}}{2} =$

$= 2 \times \frac{2\cos\alpha}{\text{sen}\,\alpha} =$

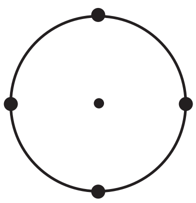
$= \frac{4}{\text{tg}\,\alpha}$.

2. Ângulo e arco generalizados e circunferência trigonométrica

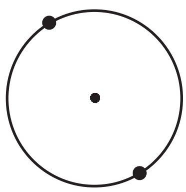
Exercícios – páginas 8 a 11

14.

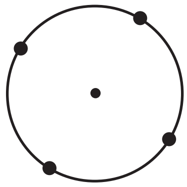
14.1. $x = 180^\circ + 90^\circ k,\ k \in \mathbb{Z}$



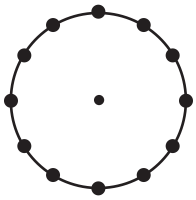
14.2. $x = -60^\circ + 180^\circ k,\ k \in \mathbb{Z}$



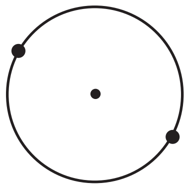
14.3. $x = -30^\circ + 90^\circ k,\ k \in \mathbb{Z}$



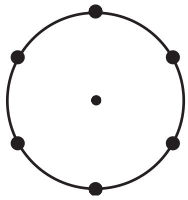
14.4. $x = 270^\circ + 30^\circ k,\ k \in \mathbb{Z}$



14.5. $x = 330^\circ + 180^\circ k,\ k \in \mathbb{Z}$



14.6. $x = -210^\circ + 60^\circ k,\ k \in \mathbb{Z}$



15.

15.1. $490^\circ = 130^\circ + 1 \times 360^\circ$

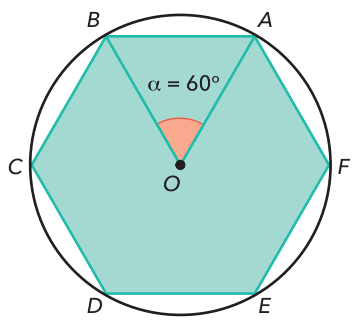
15.2. $-1550^\circ = -110^\circ - 4 \times 360^\circ$

15.3. $1755^\circ = 315^\circ + 4 \times 360^\circ$

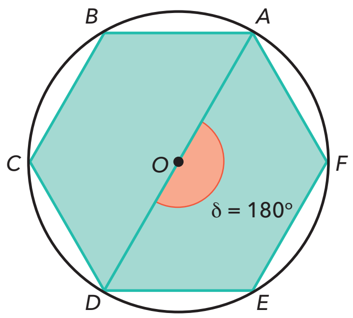
15.4. $-750^\circ = -30^\circ - 2 \times 360^\circ$

16.

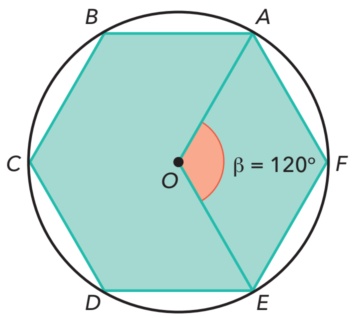
16.1. a) Ponto B



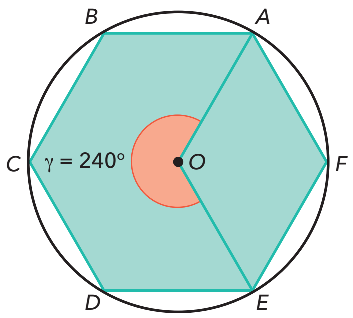
b) Ponto D



c) Ponto E



d) Ponto E



16.2. 180°, –180° e 540°, por exemplo.

16.3. a) $480° - 360° = 120°$; ponto C.
b) $-420° + 360° = -60°$; ponto F.
c) Ponto B.
d) Ponto C.

17.

17.1. $790° = 70° + 2 \times 360°$
O ângulo pertence ao primeiro quadrante.

17.2. $2770° = 250° + 7 \times 360°$
O ângulo pertence ao terceiro quadrante.

17.3. $-1730° = -290° - 4 \times 360°$
O ângulo pertence ao primeiro quadrante.

17.4. $-2920° = -40 - 8 \times 360°$
O ângulo pertence ao quarto quadrante.

18.

18.1. $\text{sen } 750° + \text{tg } 390° \times \cos 750° - \text{sen } 1125° \times \text{tg } 765° =$
$= \text{sen}(30° + 2 \times 360°) + \text{tg}(30° + 360°) \times$
$\times \cos(30° + 2 \times 360°) - \text{sen}(45° + 3 \times 360°) \times$
$\times \text{tg}(45° + 2 \times 360°) =$
$= \text{sen } 30° + \text{tg } 30° \times \cos 30° - \text{sen } 45° \times \text{tg } 45° =$
$= \frac{1}{2} + \frac{\sqrt{3}}{3} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times 1 =$
$= \frac{1}{2} + \frac{1}{2} - \frac{\sqrt{2}}{2} =$
$= 1 - \frac{\sqrt{2}}{2}$

18.2. $\cos 720° + \text{tg } 1125° - \text{sen } 450° =$
$= \cos(0° + 2 \times 360°) + \text{tg}(45° + 3 \times 360°) -$
$- \text{sen}(90° + 360°) =$
$= \cos 0° + \text{tg } 45° - \text{sen } 90° =$
$= 1 + 1 - 1 =$
$= 1$

18.3. $\text{tg } 1110° \times \text{tg } 780° - \cos 900° + \text{sen } 540° =$
$= \text{tg}(30° + 3 \times 360°) \times \text{tg}(60° + 2 \times 360°) -$
$- \cos(180° + 2 \times 360°) + \text{sen}(180° + 360°) =$
$= \text{tg } 30° \times \text{tg } 60° - \cos 180° + \text{sen } 180° =$
$= \frac{\sqrt{3}}{3} \times \sqrt{3} - (-1) + 0 =$
$= 1 + 1 =$
$= 2$

18.4. $\text{sen}^2 395° - \text{sen } 630° \times \cos 1500° + \cos^2 1115° =$
$= \text{sen}^2(35° + 360°) - \text{sen}(270° + 360°) \times$
$\times \cos(60° + 4 \times 360°) + \cos^2(35° + 3 \times 360°) =$
$= \text{sen}^2 35° - \text{sen } 270° \times \cos 60° + \cos^2 35° =$
$= 1 - (-1) \times \frac{1}{2} =$
$= 1 + \frac{1}{2} =$
$= \frac{3}{2}$

19.

19.1. Para todo $\alpha \in \mathbb{R}$,
$-1 \leq \text{sen } \alpha \leq 1 \Leftrightarrow -4 \leq 4 \text{ sen } \alpha \leq 4 \Leftrightarrow -5 \leq 4 \text{ sen } \alpha \leq 3$
A expressão pode tomar os valores do intervalo $[-5, 3]$.

19.2. Para todo $\alpha \in \mathbb{R}$,
$-1 \leq \cos \alpha \leq 1 \Leftrightarrow -3 \leq 3 \cos \alpha \leq 3$
$\Leftrightarrow -1 \leq 2 + 3 \cos \alpha \leq 5$
A expressão pode tomar os valores do intervalo $[-1, 5]$.

19.3. Para todo $\alpha \in \mathbb{R}$,
$-1 \leq \text{sen } \alpha \leq 1 \Leftrightarrow 3 \geq -3 \text{ sen } \alpha \geq -3$
$\Leftrightarrow 4 \geq 1 - 3 \text{ sen } \alpha \geq -2$
A expressão pode tomar os valores do intervalo $[-2, 4]$.

19.4. Para todo $\alpha \in \mathbb{R}$,
$-1 \leq \cos \alpha \leq 1 \Leftrightarrow -2 \leq 2 \cos \alpha \leq 2$
$\Leftrightarrow -1 \leq 2 \cos \alpha + 1 \leq 3$
$\Leftrightarrow -\frac{1}{3} \leq \frac{2 \cos \alpha + 1}{3} \leq 1$
A expressão pode tomar os valores do intervalo $\left[-\frac{1}{3}, 1\right]$.

20.

20.1. Para todo $x \in \mathbb{R}$,
$-1 \leq \text{sen } x \leq 1 \Leftrightarrow -1 \leq \frac{5 - 3k}{2} \leq 1 \Leftrightarrow -2 \leq 5 - 3k \leq 2$
$\Leftrightarrow -7 \leq -3k \leq -3 \Leftrightarrow 7 \geq 3k \geq 3 \Leftrightarrow \frac{7}{3} \geq k \geq 1$
Logo, $k \in \left[1, \frac{7}{3}\right]$.

20.2. Para todo $x \in]180°, 270°[$,
$-1 \leq \text{sen}\, x < 0 \Leftrightarrow -3 \leq 3\,\text{sen}\, x < 0 \Leftrightarrow -3 \leq 4k + 7 < 0$
$\Leftrightarrow -10 \leq 4k < -7 \Leftrightarrow -\frac{5}{2} \leq k < -\frac{7}{4}$
Logo, $k \in \left[-\frac{5}{2}, -\frac{7}{4}\right[$.

20.3. Para todo $x \in [45°, 135°[$,
$-\frac{\sqrt{2}}{2} < \cos x \leq \frac{\sqrt{2}}{2} \Leftrightarrow -3\sqrt{2} < 6\cos x \leq 3\sqrt{2}$
$\Leftrightarrow -3\sqrt{2} < 2k - \sqrt{2} \leq 3\sqrt{2} \Leftrightarrow -2\sqrt{2} < 2k \leq 4\sqrt{2}$
$\Leftrightarrow -\sqrt{2} < k \leq 2\sqrt{2}$
Logo, $k \in]-\sqrt{2}, 2\sqrt{2}]$.

20.4. Para todo $x \in]-60°, 30°[$,
$\frac{1}{2} < \cos x \leq 1 \Leftrightarrow \frac{1}{2} < 5k - 3 \leq 1 \Leftrightarrow \frac{7}{2} < 5k \leq 4$
$\Leftrightarrow \frac{7}{10} < k \leq \frac{4}{5}$
Logo, $k \in \left]\frac{7}{10}, \frac{4}{5}\right]$.

21. Como a ordenada de B é $\sqrt{8}$, então $\text{tg}\, \alpha = \sqrt{8}$.
Então, $1 + \text{tg}^2\alpha = \frac{1}{\cos^2\alpha} \Leftrightarrow 1 + (\sqrt{8})^2 = \frac{1}{\cos^2\alpha}$
$\Leftrightarrow 9 = \frac{1}{\cos^2\alpha}$
$\Leftrightarrow \cos^2\alpha = \frac{1}{9}$
$\Leftrightarrow \cos\alpha = \pm\frac{1}{3}$
Como α pertence ao primeiro quadrante, então $\cos\alpha = \frac{1}{3}$.
Como a ordenada de C é $-\frac{4}{5}$, então $\text{sen}\,\beta = -\frac{4}{5}$.
Pela fórmula fundamental da trigonometria,
$\text{sen}^2\beta + \cos^2\beta = 1 \Leftrightarrow \left(-\frac{4}{5}\right)^2 + \cos^2\beta = 1$
$\Leftrightarrow \cos\beta = 1 - \frac{16}{25}$
$\Leftrightarrow \cos^2\beta = \frac{9}{25}$
$\Leftrightarrow \cos\beta = \pm\frac{3}{5}$
Como β pertence ao terceiro quadrante, então $\cos\beta = -\frac{3}{5}$.
Tem-se então que $\text{tg}\,\beta = \frac{\text{sen}\,\beta}{\cos\beta} = \frac{-\frac{4}{5}}{-\frac{3}{5}} = \frac{4}{3}$.
Assim,
$\cos\alpha - 3\,\text{sen}\,\beta + \frac{\text{tg}\,\alpha}{\sqrt{2}\,\text{tg}\,\beta} = \frac{1}{3} - 3 \times \left(-\frac{4}{5}\right) + \frac{\sqrt{8}}{\sqrt{2} \times \frac{4}{3}} =$
$= \frac{127}{30}$

22.

22.1. As coordenadas do ponto B são $(\cos\alpha, \text{sen}\,\alpha)$.
Logo, $\overline{BC} = \text{sen}\,\alpha$, $\overline{OC} = \cos\alpha$ e $\overline{CD} = 2 - \cos\alpha$.
Assim, $A_{[BCD]} = \frac{\overline{CD} \times \overline{BC}}{2} =$
$= \frac{(2 - \cos\alpha) \times \text{sen}\,\alpha}{2} =$
$= \text{sen}\,\alpha \left(\frac{2 - \cos\alpha}{2}\right) =$
$= \text{sen}\,\alpha \left(1 - \frac{2 - \cos\alpha}{2}\right) =$
$= A(\alpha)$

22.2. Tem-se que $\text{tg}\,\alpha = \frac{\sqrt{3}}{2}$. Então,
$1 + \text{tg}^2\alpha = \frac{1}{\cos^2\alpha} \Leftrightarrow 1 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{\cos^2\alpha}$
$\Leftrightarrow 1 + \frac{3}{4} = \frac{1}{\cos^2\alpha}$
$\Leftrightarrow \frac{7}{4} = \frac{1}{\cos^2\alpha}$
$\Leftrightarrow \cos^2\alpha = \frac{4}{7}$
$\Leftrightarrow \cos\alpha = \pm\frac{2}{\sqrt{7}}$
$\Leftrightarrow \cos\alpha = \pm\frac{2\sqrt{7}}{7}$
Como α pertence ao primeiro quadrante, então $\cos\alpha = \frac{2\sqrt{7}}{7}$.
Pela fórmula fundamental da trigonometria,
$\text{sen}^2\alpha + \cos^2\alpha = 1 \Leftrightarrow \text{sen}^2\alpha + \frac{4}{7} = 1$
$\Leftrightarrow \text{sen}^2\alpha = \frac{3}{7}$
$\Leftrightarrow \text{sen}\,\alpha = \pm\sqrt{\frac{3}{7}}$
$\Leftrightarrow \text{sen}\,\alpha = \pm\frac{\sqrt{21}}{7}$
Como α pertence ao primeiro quadrante, então $\text{sen}\,\alpha = \frac{\sqrt{21}}{7}$.
Assim, $A(\alpha) = \text{sen}\,\alpha\left(1 - \frac{\cos\alpha}{2}\right) =$
$= \frac{\sqrt{21}}{7}\left(1 - \frac{\frac{2\sqrt{7}}{7}}{2}\right) =$
$= \frac{\sqrt{21}}{7} - \frac{7\sqrt{3}}{49} =$
$= \frac{\sqrt{21}}{7} - \frac{\sqrt{3}}{7}$

23.

23.1. $\text{sen}^4\alpha - \cos^4\alpha =$
$= (\text{sen}^2 x + \cos^2 x) \times (\text{sen}^2\alpha - \cos^2\alpha) =$
$= 1 \times (\text{sen}^2\alpha - \cos^2\alpha) =$
$= \text{sen}^2\alpha - \cos^2\alpha$

23.2. $(\text{sen}\,\alpha + \cos\alpha)^2 + (\text{sen}\,\alpha - \cos\alpha)^2 =$
$= \text{sen}^2\alpha + 2\,\text{sen}\,\alpha\cos\alpha + \cos^2\alpha + \text{sen}^2\alpha -$
$- 2\,\text{sen}\,\alpha\cos\alpha + \cos^2\alpha =$
$= 2(\text{sen}^2\alpha + \cos^2\alpha) =$
$= 2$

23.3. $\frac{\text{sen}\,\alpha}{1 + \cos\alpha} + \frac{1 - \cos\alpha}{\text{sen}\,\alpha} =$
$= \frac{\text{sen}\,\alpha}{1 + \cos\alpha} \times \frac{1 - \cos\alpha}{1 - \cos\alpha} + \frac{1 - \cos\alpha}{\text{sen}\,\alpha} \times \frac{\text{sen}\,\alpha}{\text{sen}\,\alpha} =$
$= \frac{\text{sen}\,\alpha - \text{sen}\,\alpha\cos\alpha}{1 - \cos^2\alpha} + \frac{\text{sen}\,\alpha + \text{sen}\,\alpha\cos\alpha}{\text{sen}^2\alpha} =$

$= \frac{\text{sen}\,\alpha - \text{sen}\,\alpha\cos\alpha}{\text{sen}^2\alpha} + \frac{\text{sen}\,\alpha + \text{sen}\,\alpha\cos\alpha}{\text{sen}^2\alpha} =$

$= \frac{2\,\text{sen}\,\alpha}{\text{sen}^2\alpha} =$

$= \frac{2}{\text{sen}\,\alpha}$

23.4. $\frac{\text{tg}\,\alpha + \text{sen}\,\alpha}{\text{tg}\,\alpha} =$

$= \frac{\frac{\text{sen}\,\alpha}{\cos\alpha} + \text{sen}\,\alpha}{\frac{\text{sen}\,\alpha}{\cos\alpha}} =$

$= \frac{\frac{\text{sen}\,\alpha + \text{sen}\,\alpha\cos\alpha}{\cos\alpha}}{\frac{\text{sen}\,\alpha}{\cos\alpha}} =$

$= \frac{\text{sen}\,\alpha + \text{sen}\,\alpha\cos\alpha}{\text{sen}\,\alpha} =$

$= \frac{\text{sen}\,\alpha(1+\cos\alpha)}{\text{sen}\,\alpha} =$

$= 1 + \cos\alpha$

23.5. $\frac{1 - 2\,\text{sen}^2\alpha}{2\cos^2\alpha - 1} =$

$= \frac{1 - \text{sen}^2\alpha - \text{sen}^2\alpha}{\cos^2\alpha + \cos^2\alpha - 1} =$

$= \frac{\cos^2\alpha - \text{sen}^2\alpha}{\cos^2\alpha - (1 - \cos^2\alpha)} =$

$= \frac{\cos^2\alpha - \text{sen}^2\alpha}{\cos^2\alpha - \text{sen}^2\alpha} =$

$= 1$

23.6. $\frac{1}{\text{sen}\,\alpha} - \frac{1}{\text{tg}\,\alpha} =$

$= \frac{1}{\text{sen}\,\alpha} - \frac{1}{\frac{\text{sen}\,\alpha}{\cos\alpha}} =$

$= \frac{1}{\text{sen}\,\alpha} - \frac{\cos\alpha}{\text{sen}\,\alpha} =$

$= \frac{1 - \cos\alpha}{\text{sen}\,\alpha}$

24.

24.1. As coordenadas do ponto A são $(\cos\alpha, \text{sen}\,\alpha)$.
Logo, $\overline{AC} = \text{sen}\,\alpha$ e $\overline{OC} = \cos\alpha$.
Então, $\overline{DE} = \overline{CD} = 1 - \cos\alpha$ e, portanto,
$\overline{OE} = 1 + 1 - \cos\alpha = 2 - \cos\alpha$.

Assim, $A_{[AOBE]} = 2 \times \frac{\overline{DE} \times \overline{AC}}{2} = (2 - \cos\alpha) \times \text{sen}\,\alpha =$
$= 2\,\text{sen}\,\alpha - \text{sen}\,\alpha\cos\alpha$.

24.2. $\overline{AC} = \text{sen}\,\alpha$ e $\overline{CE} = 2 - 2\cos\alpha$, pela alínea anterior.
Pelo teorema de Pitágoras, $\overline{AE}^2 = \overline{AC}^2 + \overline{CE}^2$.
Assim, $d^2 = (\text{sen}\,\alpha)^2 + (2 - 2\cos\alpha)^2$
$\Leftrightarrow d^2 = \text{sen}^2\alpha + 4 - 8\cos\alpha + 4\cos^2\alpha$
$\Leftrightarrow d^2 = \text{sen}^2\alpha + \cos^2\alpha + 4 - 8\cos\alpha + 3\cos^2\alpha$
$\Leftrightarrow d^2 = 5 - 8\cos\alpha + 3\cos^2\alpha$
Logo, $d = \sqrt{3\cos^2\alpha - 8\cos\alpha + 5}$.

24.3. $\overline{OC} = \overline{CE} \Leftrightarrow \cos\alpha = 2 - 2\cos\alpha \Leftrightarrow 3\cos\alpha = 2$

$\Leftrightarrow \cos\alpha = \frac{2}{3}$

Logo, $\alpha = \cos^{-1}\left(\frac{2}{3}\right) \approx 48{,}19^\circ$.

24.4. $d = \frac{\sqrt{7}}{2} \Leftrightarrow \sqrt{3\cos^2\alpha - 8\cos\alpha + 5} = \frac{\sqrt{7}}{2}$

$\Leftrightarrow 3\cos^2\alpha - 8\cos\alpha + 5 = \frac{7}{4}$

$\Leftrightarrow 3\cos^2\alpha - 8\cos\alpha + \frac{13}{4} = 0$

$\Leftrightarrow \cos\alpha = \frac{8 \pm \sqrt{8^2 - 4 \times 3 \times \frac{13}{4}}}{6}$

$\Leftrightarrow \cos\alpha = \frac{8 \pm 5}{6}$

$\Leftrightarrow \cos\alpha = \frac{13}{6} \vee \cos\alpha = \frac{1}{2}$

Como $\cos\alpha = \frac{13}{6}$ é impossível, então $\cos\alpha = \frac{1}{2}$ e, uma vez que $\alpha \in [0^\circ, 90^\circ]$, tem-se que $\alpha = 60^\circ$.

3. Radiano e redução ao primeiro quadrante

Exercícios – páginas 12 a 14

25.

25.1. $\frac{\pi}{180} = \frac{x}{70} \Leftrightarrow x = \frac{70\pi}{180} \Leftrightarrow x = \frac{7\pi}{18}$

Logo, $\frac{7\pi}{18}$ é a medida em radianos de um ângulo de 70°.

25.2. $\frac{\pi}{180} = \frac{x}{135} \Leftrightarrow x = \frac{135\pi}{180} \Leftrightarrow x = \frac{3\pi}{4}$

Logo, $\frac{3\pi}{4}$ é a medida em radianos de um ângulo de 135°.

25.3. $\frac{\pi}{180} = \frac{x}{252} \Leftrightarrow x = \frac{252\pi}{180} \Leftrightarrow x = \frac{7\pi}{5}$

Logo, $\frac{7\pi}{5}$ é a medida em radianos de um ângulo de 252°.

25.4. $\frac{\pi}{180} = \frac{x}{300} \Leftrightarrow x = \frac{300\pi}{180} \Leftrightarrow x = \frac{5\pi}{3}$

Logo, $\frac{5\pi}{3}$ é a medida em radianos de um ângulo de 300°.

26.

26.1. $\frac{\pi}{180} = \frac{\frac{3\pi}{5}}{x} \Leftrightarrow x = 180 \times \frac{3\pi}{5} \Leftrightarrow x = 108$

Logo, 108° é a medida de um ângulo de $\frac{3\pi}{5}$ rad.

26.2. $\frac{\pi}{180} = \frac{\frac{5\pi}{8}}{x} \Leftrightarrow x = 180 \times \frac{5\pi}{8} \Leftrightarrow x = 112{,}5$

Logo, $112{,}5^\circ$ é a medida de um ângulo de $\frac{5\pi}{8}$ rad.

26.3. $\frac{\pi}{180} = \frac{\frac{7\pi}{10}}{x} \Leftrightarrow x = 180 \times \frac{7\pi}{10} \Leftrightarrow x = 126$

Logo, 126° é a medida de um ângulo de $\frac{7\pi}{10}$ rad.

26.4. $\frac{\pi}{180} = \frac{\frac{2\pi}{9}}{x} \Leftrightarrow x = 180 \times \frac{2\pi}{9} \Leftrightarrow x = 40$

Logo, 40° é a medida de um ângulo de $\frac{2\pi}{9}$ rad.

27.

27.1. a) $\frac{360}{2\pi \times 3} = \frac{30}{x} \Leftrightarrow x = \frac{6\pi \times 30}{360} \Leftrightarrow x = \frac{\pi}{2}$

Logo, o comprimento do arco AB é $\frac{\pi}{2}$ cm.

b) $\dfrac{2\pi}{2\pi \times 6} = \dfrac{\frac{\pi}{5}}{x} \Leftrightarrow x = \dfrac{12\pi \times \frac{\pi}{5}}{2\pi} \Leftrightarrow x = \dfrac{6\pi}{5}$

Logo, o comprimento do arco AB é $\dfrac{6\pi}{5}$ cm.

27.2. a) $\dfrac{360}{\pi \times 2^2} = \dfrac{70}{x} \Leftrightarrow x = \dfrac{4\pi \times 70}{360} \Leftrightarrow x = \dfrac{7\pi}{9}$

Logo, a área do setor circular AOB é $\dfrac{7\pi}{9}$ cm^2.

b) $\dfrac{2\pi}{\pi \times 4^2} = \dfrac{\frac{\pi}{6}}{x} \Leftrightarrow x = \dfrac{16\pi \times \frac{\pi}{6}}{2\pi} \Leftrightarrow x = \dfrac{4\pi}{3}$

Logo, a área do setor circular AOB é $\dfrac{4\pi}{3}$ cm^2.

28. $\dfrac{2\pi}{\pi \times r^2} = \dfrac{\frac{3\pi}{10}}{15\pi} \Leftrightarrow \dfrac{2}{r^2} = \dfrac{3}{150} \Leftrightarrow r^2 = \dfrac{2 \times 150}{3} \Leftrightarrow r^2 = 100$

Logo, $r = 10$, ou seja, o raio da circunferência mede 10 cm.

29. $\dfrac{2\pi}{2\pi \times r} = \dfrac{\alpha}{\frac{2\pi}{3}} \Leftrightarrow \dfrac{1}{r} = \dfrac{3\alpha}{2\pi} \Leftrightarrow \alpha = \dfrac{2\pi}{3r}$

$\dfrac{2\pi}{\pi \times r^2} = \dfrac{\alpha}{3\pi} \Leftrightarrow \alpha = \dfrac{6\pi}{r^2}$

$\Leftrightarrow \dfrac{2\pi}{3r} = \dfrac{6\pi}{r^2}$

$\Leftrightarrow 2r^2 = 18r$

$\Leftrightarrow r^2 - 9r = 0$

$\Leftrightarrow r(r - 9) = 0$

$\Leftrightarrow r = 0 \ \vee \ r = 9$

Logo, $r = 9$ cm.

$\alpha = \dfrac{2\pi}{3r} = \dfrac{2\pi}{27}$

Logo, $\alpha = \dfrac{2\pi}{27}$ rad.

30.

30.1. $\text{sen}\left(-\dfrac{5\pi}{3}\right) - \cos\dfrac{5\pi}{6} + \text{tg}\,\dfrac{2\pi}{3} - \text{sen}\,\dfrac{7\pi}{6} =$

$= \text{sen}\left(-\dfrac{5\pi}{3} + 2\pi\right) - \cos\left(\pi - \dfrac{\pi}{6}\right) + \text{tg}\left(\pi - \dfrac{\pi}{3}\right) -$

$- \text{sen}\left(\pi + \dfrac{\pi}{6}\right) =$

$= \text{sen}\,\dfrac{\pi}{3} + \cos\dfrac{\pi}{6} - \text{tg}\,\dfrac{\pi}{3} + \text{sen}\,\dfrac{\pi}{6} =$

$= \dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{3}}{2} - \sqrt{3} + \dfrac{1}{2} =$

$= \dfrac{1}{2}$

30.2. $\cos\dfrac{11\pi}{6} \times \text{tg}\left(-\dfrac{3\pi}{4}\right) - \text{sen}\left(-\dfrac{4\pi}{3}\right) + \cos\dfrac{3\pi}{4} =$

$= \cos\left(2\pi - \dfrac{\pi}{6}\right) \times \left(-\text{tg}\,\dfrac{3\pi}{4}\right) + \text{sen}\,\dfrac{4\pi}{3} + \cos\left(\pi - \dfrac{\pi}{4}\right) =$

$= \cos\left(-\dfrac{\pi}{6}\right) \times \left(-\text{tg}\left(\pi - \dfrac{\pi}{4}\right)\right) + \text{sen}\left(\pi + \dfrac{\pi}{3}\right) - \cos\dfrac{\pi}{4} =$

$= \cos\dfrac{\pi}{6} \times \text{tg}\,\dfrac{\pi}{4} - \text{sen}\,\dfrac{\pi}{3} - \cos\dfrac{\pi}{4} =$

$= \dfrac{\sqrt{3}}{2} \times 1 - \dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{2}}{2} =$

$= -\dfrac{\sqrt{2}}{2}$

30.3. $\text{tg}\,\dfrac{29\pi}{3} + \text{sen}\,\dfrac{27\pi}{4} + \text{sen}\left(-\dfrac{31\pi}{6}\right) + \cos\dfrac{40\pi}{3} =$

$= \text{tg}\left(9\pi + \dfrac{2\pi}{3}\right) + \text{sen}\left(6\pi + \dfrac{3\pi}{4}\right) - \text{sen}\left(\dfrac{31\pi}{6}\right) +$

$+ \cos\left(13\pi + \dfrac{\pi}{3}\right) =$

$= \text{tg}\,\dfrac{2\pi}{3} + \text{sen}\,\dfrac{3\pi}{4} - \text{sen}\left(5\pi + \dfrac{\pi}{6}\right) + \cos\left(\pi + \dfrac{\pi}{3}\right) =$

$= \text{tg}\left(\pi - \dfrac{\pi}{3}\right) + \text{sen}\left(\pi - \dfrac{\pi}{4}\right) - \text{sen}\left(\pi + \dfrac{\pi}{6}\right) - \cos\dfrac{\pi}{3} =$

$= -\text{tg}\,\dfrac{\pi}{3} + \text{sen}\,\dfrac{\pi}{4} + \text{sen}\,\dfrac{\pi}{6} - \cos\dfrac{\pi}{3} =$

$= -\sqrt{3} + \dfrac{\sqrt{2}}{2} + \dfrac{1}{2} - \dfrac{1}{2} =$

$= \dfrac{-2\sqrt{3} + \sqrt{2}}{2}$

30.4. $\cos\dfrac{150\pi}{4} - \text{tg}\,\dfrac{123\pi}{3} + \cos\dfrac{139\pi}{6} \times \text{tg}\,\dfrac{100\pi}{3} =$

$= \cos\left(37\pi + \dfrac{\pi}{2}\right) - \text{tg}(41\pi) + \cos\left(23\pi + \dfrac{\pi}{6}\right) \times$

$\times\, \text{tg}\left(33\pi + \dfrac{\pi}{3}\right) =$

$= \cos\left(\pi + \dfrac{\pi}{2}\right) - 0 + \cos\left(\pi + \dfrac{\pi}{6}\right) \times \text{tg}\,\dfrac{\pi}{3} =$

$= 0 - \cos\dfrac{\pi}{6} \times \text{tg}\,\dfrac{\pi}{3} =$

$= -\dfrac{\sqrt{3}}{2} \times \sqrt{3} =$

$= -\dfrac{3}{2}$

31.

31.1. $\cos\left(\dfrac{\pi}{2} - x\right) + \cos\left(\dfrac{\pi}{2} + x\right) - \text{sen}(\pi + x) =$

$= \text{sen}\,x - \text{sen}\,x + \text{sen}\,x =$

$= \text{sen}\,x$

31.2. $\text{sen}\left(-\dfrac{3\pi}{2} + x\right) + \text{tg}(5\pi + x) - \text{sen}\left(-\dfrac{\pi}{2} - x\right) =$

$= -\text{sen}\left(\pi + \dfrac{\pi}{2} - x\right) + \text{tg}\,x + \text{sen}\left(\dfrac{\pi}{2} + x\right) =$

$= \text{sen}\left(\dfrac{\pi}{2} - x\right) + \text{tg}\,x + \cos x =$

$= \cos x + \text{tg}\,x + \cos x =$

$= 2\cos x + \text{tg}\,x$

31.3. $\text{tg}\left(x - \dfrac{\pi}{2}\right) + \text{tg}(3\pi - x) + \text{tg}\left(\dfrac{7\pi}{2} - x\right) =$

$= -\text{tg}\left(\dfrac{\pi}{2} - x\right) + \text{tg}(\pi - x) + \text{tg}\left(3\pi + \dfrac{\pi}{2} - x\right) =$

$= -\text{tg}\left(\dfrac{\pi}{2} - x\right) - \text{tg}\,x + \text{tg}\left(\dfrac{\pi}{2} - x\right) =$

$= -\text{tg}\,x$

31.4. $\text{sen}\left(\dfrac{11\pi}{2} - x\right) - \cos\left(\dfrac{23\pi}{2} - x\right) - \text{sen}(29\pi - x) =$

$= \text{sen}\left(5\pi + \dfrac{\pi}{2} - x\right) - \cos\left(11\pi + \dfrac{\pi}{2} - x\right) - \text{sen}(\pi - x) =$

$= \text{sen}\left(\pi + \dfrac{\pi}{2} - x\right) - \cos\left(\pi + \dfrac{\pi}{2} - x\right) - \text{sen}\,x =$

$= -\text{sen}\left(\dfrac{\pi}{2} - x\right) + \cos\left(\dfrac{\pi}{2} - x\right) - \text{sen}\,x =$

$= -\cos x + \operatorname{sen} x - \operatorname{sen} x =$
$= -\cos x$

31.5. $\operatorname{tg}(x - \pi) + \cos\left(-\frac{17\pi}{2} + x\right) + \operatorname{tg}(x - 12\pi) =$

$= -\operatorname{tg}(\pi - x) + \cos\left(8\pi + \frac{\pi}{2} - x\right) - \operatorname{tg}(12\pi - x) =$

$= \operatorname{tg} x + \cos\left(\frac{\pi}{2} - x\right) - \operatorname{tg}(-x) =$

$= \operatorname{tg} x + \operatorname{sen} x + \operatorname{tg} x =$
$= \operatorname{sen} x + 2 \operatorname{tg} x$

31.6. $\operatorname{sen}\left(\frac{32\pi}{2} + x\right) - \operatorname{sen}\left(x - \frac{51\pi}{2}\right) - \cos(-x - 125\pi) =$

$= \operatorname{sen}(16\pi + x) + \operatorname{sen}\left(\frac{51\pi}{2} - x\right) - \cos(125\pi + x) =$

$= \operatorname{sen} x + \operatorname{sen}\left(25\pi + \frac{\pi}{2} - x\right) - \cos(\pi + x) =$

$= \operatorname{sen} x + \operatorname{sen}\left(\pi + \frac{\pi}{2} - x\right) + \cos x =$

$= \operatorname{sen} x - \operatorname{sen}\left(\frac{\pi}{2} - x\right) + \cos x =$

$= \operatorname{sen} x - \cos x + \cos x =$
$= \operatorname{sen} x$

32.

32.1. $\cos 225^\circ + \operatorname{sen} 120^\circ + \cos 330^\circ - \operatorname{sen} 315^\circ =$
$= \cos(180^\circ + 45^\circ) + \operatorname{sen}(180^\circ - 60^\circ) +$
$+ \cos(360^\circ - 30^\circ) - \operatorname{sen}(360^\circ - 45^\circ) =$
$= -\cos 45^\circ + \operatorname{sen} 60^\circ + \cos(-30^\circ) - \operatorname{sen}(-45^\circ) =$
$= -\cos 45^\circ + \operatorname{sen} 60^\circ + \cos 30^\circ + \operatorname{sen} 45^\circ =$
$= -\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} =$
$= \sqrt{3}$

32.2. $\operatorname{sen} 210^\circ + \cos 150^\circ \times \operatorname{tg} 135^\circ - \cos 300^\circ =$
$= \operatorname{sen}(180^\circ + 30^\circ) + \cos(180^\circ - 30^\circ) \times$
$\times \operatorname{tg}(180^\circ - 45^\circ) - \cos(360^\circ - 60^\circ) =$
$= -\operatorname{sen} 30^\circ - \cos 30^\circ \times (-\operatorname{tg} 45^\circ) - \cos(-60^\circ) =$
$= -\operatorname{sen} 30^\circ + \cos 30^\circ \times \operatorname{tg} 45^\circ - \cos 60^\circ =$
$= -\frac{1}{2} + \frac{\sqrt{3}}{2} \times 1 - \frac{1}{2} =$
$= \frac{\sqrt{3}}{2} - 1$

32.3. $\cos 1305^\circ \times \operatorname{sen} 1590^\circ \times \operatorname{tg} 780^\circ =$
$= \cos(3 \times 360^\circ + 225^\circ) \times \operatorname{sen}(4 \times 360^\circ + 150^\circ) \times$
$\times \operatorname{tg}(2 \times 360^\circ + 60^\circ) =$
$= \cos 225^\circ \times \operatorname{sen} 150^\circ \times \operatorname{tg} 60^\circ =$
$= \cos(180^\circ + 45^\circ) \times \operatorname{sen}(180^\circ - 30^\circ) \times \operatorname{tg} 60^\circ =$
$= -\cos 45^\circ \times \operatorname{sen} 30^\circ \times \operatorname{tg} 60^\circ =$
$= -\frac{\sqrt{2}}{2} \times \frac{1}{2} \times \sqrt{3} =$
$= -\frac{\sqrt{6}}{4}$

32.4. $\operatorname{tg} 1575^\circ + \cos 765^\circ \times \operatorname{sen} 1395^\circ =$
$= \operatorname{tg}(4 \times 360^\circ + 135^\circ) + \cos(2 \times 360^\circ + 45^\circ) \times$
$\times \operatorname{sen}(3 \times 360^\circ + 315^\circ) =$
$= \operatorname{tg} 135^\circ + \cos 45^\circ \times \operatorname{sen} 315^\circ =$
$= \operatorname{tg}(180^\circ - 45^\circ) + \cos 45^\circ \times \operatorname{sen}(360^\circ - 45^\circ) =$
$= -\operatorname{tg} 45^\circ + \cos 45 \times \operatorname{sen}(-45^\circ) =$
$= -\operatorname{tg} 45^\circ - \cos 45^\circ \times \operatorname{sen} 45^\circ =$

$= -1 - \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} =$

$= -1 - \frac{1}{2} =$

$= -\frac{3}{2}$

33.

33.1. $\operatorname{sen}(450^\circ + x) - \operatorname{tg}(900^\circ + x) + \operatorname{sen}(-90^\circ - x) =$
$= \operatorname{sen}(360^\circ + 90^\circ + x) - \operatorname{tg}(2 \times 360^\circ + 180^\circ + x) -$
$- \operatorname{sen}(90^\circ + x) =$
$= \operatorname{sen}(90^\circ + x) - \operatorname{tg} x - \operatorname{sen}(90^\circ + x) =$
$= -\operatorname{tg} x$

33.2. $\cos(x - 90^\circ) + \operatorname{sen}(180^\circ + x) + \cos(180^\circ - x) =$
$= \cos(90^\circ - x) - \operatorname{sen} x - \cos x =$
$= \operatorname{sen} x - \operatorname{sen} x - \cos x =$
$= -\cos x$

33.3. $\operatorname{tg}(450^\circ - x) - \operatorname{tg}(180^\circ - x) - \operatorname{sen}(270^\circ - x) -$
$- \operatorname{tg}(-270^\circ - x) =$
$= \operatorname{tg}(360^\circ + 90^\circ - x) + \operatorname{tg} x - \operatorname{sen}(180^\circ + 90^\circ - x) -$
$- \operatorname{tg}(90^\circ - x) =$
$= \operatorname{tg}(90^\circ - x) + \operatorname{tg} x + \operatorname{sen}(90^\circ - x) - \operatorname{tg}(90^\circ - x) =$
$= \operatorname{tg} x + \cos x$

33.4. $\operatorname{tg}(-x - 90^\circ) + \operatorname{tg}(450^\circ + x) + \operatorname{tg}(x - 180^\circ) =$
$= -\operatorname{tg}(90^\circ + x) + \operatorname{tg}(360^\circ + 90^\circ + x) - \operatorname{tg}(180^\circ - x) =$
$= -\operatorname{tg}(90^\circ + x) + \operatorname{tg}(90^\circ + x) + \operatorname{tg} x =$
$= \operatorname{tg} x$

33.5. $-\cos(270^\circ + x) + \operatorname{sen}(x - 180^\circ) - \cos(-x - 180^\circ) =$
$= -\cos(180^\circ + 90^\circ + x) - \operatorname{sen}(180^\circ - x) - \cos(180^\circ + x) =$
$= \cos(90^\circ + x) - \operatorname{sen} x + \cos x =$
$= -\operatorname{sen} x - \operatorname{sen} x + \cos x =$
$= -2 \operatorname{sen} x + \cos x$

34.

34.1. $\cos\left(\frac{3\pi}{2} - \alpha\right) + \cos\left(\frac{5\pi}{2} + \alpha\right) - \operatorname{sen}(10\pi + \alpha) =$

$= \cos\left(\pi + \frac{\pi}{2} - \alpha\right) + \cos\left(2\pi + \frac{\pi}{2} + \alpha\right) - \operatorname{sen} \alpha =$

$= -\cos\left(\frac{\pi}{2} - \alpha\right) + \cos\left(\frac{\pi}{2} + \alpha\right) - \operatorname{sen} \alpha =$

$= -\operatorname{sen} \alpha - \operatorname{sen} \alpha - \operatorname{sen} \alpha =$
$= -3 \operatorname{sen} \alpha$

Uma vez que $\cos\left(\frac{\pi}{2} - \alpha\right) = -\frac{1}{5} \Leftrightarrow \operatorname{sen} \alpha = -\frac{1}{5}$,

então $-3 \operatorname{sen} \alpha = -3 \times \left(-\frac{1}{5}\right) = \frac{3}{5}$.

34.2. $\operatorname{tg}(\alpha - \pi) + \operatorname{sen}\left(\frac{\pi}{2} + \alpha\right) = -\operatorname{tg}(\pi - \alpha) + \cos \alpha =$
$= \operatorname{tg} \alpha + \cos \alpha$

Ora, $\cos\left(\frac{\pi}{2} - \alpha\right) = -\frac{1}{5} \Leftrightarrow \operatorname{sen} \alpha = -\frac{1}{5}$.

$\operatorname{sen}^2\alpha + \cos^2\alpha = 1 \Leftrightarrow \cos^2\alpha = 1 - \left(-\frac{1}{5}\right)^2$

$\Leftrightarrow \cos^2\alpha = \frac{24}{25}$

Como $\alpha \in \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[$ e $\operatorname{sen} \alpha < 0$, então α pertence ao terceiro quadrante, pelo que $\cos \alpha = -\sqrt{\frac{24}{25}} =$
$= -\frac{2\sqrt{6}}{5}$.

$\text{tg } \alpha = \frac{\text{sen } \alpha}{\cos \alpha} = \frac{-\frac{1}{5}}{-\frac{2\sqrt{6}}{5}} = \frac{1}{2\sqrt{6}} = \frac{\sqrt{6}}{12}$

Logo, $\text{tg } \alpha + \cos \alpha = \frac{\sqrt{6}}{12} - \frac{2\sqrt{6}}{5} = \frac{5\sqrt{6} - 24\sqrt{6}}{60} =$

$= -\frac{19\sqrt{6}}{60}$.

35.

35.1. $\overline{AB} = 7$

$\text{tg } x = \frac{\overline{BC}}{\overline{AB}} \Leftrightarrow \overline{BC} = 7 \text{ tg } x$

$\cos x = \frac{\overline{AB}}{\overline{AC}} \Leftrightarrow \overline{AC} = \frac{7}{\cos x}$

Logo, o perímetro de [ABC] é dado, em função de x, por $7 + 7 \text{ tg } x + \frac{7}{\cos x} = 7\left(1 + \text{tg } x + \frac{1}{\cos x}\right) = P(x)$.

35.2. $P\left(\frac{\pi}{3}\right) = 7\left(1 + \text{tg } \frac{\pi}{3} + \frac{1}{\cos \frac{\pi}{3}}\right) = 7\left(1 + \sqrt{3} + \frac{1}{\frac{1}{2}}\right) =$

$= 7(1 + \sqrt{3} + 2) = 7(3 + \sqrt{3})$ u.c.

35.3. $\overline{AB} = 7$

$\text{tg } x = \frac{\overline{BC}}{\overline{AB}} \Leftrightarrow \overline{BC} = 7 \text{ tg } x$

$\overline{AD} = 3$

$\overline{DB} = 7 - 3 = 4$

$\text{tg } x = \frac{\overline{DE}}{\overline{AD}} \Leftrightarrow \overline{DE} = 3 \text{ tg } x$

Logo, a área do polígono [BCED] é dada, em função de x, por $\frac{7 \text{ tg } x + 3 \text{ tg } x}{2} \times 4 = 5 \text{ tg } x \times 4 = 20 \text{ tg } x = A(x)$.

35.4. $\cos\left(\frac{\pi}{2} + \alpha\right) = -\frac{1}{3} \Leftrightarrow -\text{sen } \alpha = -\frac{1}{3} \Leftrightarrow \text{sen } \alpha = \frac{1}{3}$

$\text{sen}^2\alpha + \cos^2\alpha = 1 \Leftrightarrow \cos^2\alpha = 1 - \left(\frac{1}{3}\right)^2 \Leftrightarrow \cos^2\alpha = \frac{8}{9}$

Uma vez que $\alpha \in \left]0, \frac{\pi}{2}\right[$, $\cos \alpha = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$.

$\text{tg } \alpha = \frac{\text{sen } \alpha}{\cos \alpha} = \frac{\frac{1}{3}}{\frac{2\sqrt{2}}{3}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$.

Logo, $A(\alpha) = 20 \text{ tg } \alpha = 20 \times \frac{\sqrt{2}}{4} = 5\sqrt{2}$ u.a.

36.

36.1. $A(\alpha) = A_{[AOB]} + A_{[OCA]}$

$A_{[AOB]} = \frac{\overline{OA} \times h}{2} = \frac{1 \times \text{sen } \alpha}{2} = \frac{\text{sen } \alpha}{2}$

$A_{[OCA]} = \frac{\overline{OA} \times \overline{AC}}{2} = \frac{1 \times (-\text{tg } \alpha)}{2} = -\frac{\text{tg } \alpha}{2}$

Como $\alpha \in \left]\frac{\pi}{2}, \pi\right[$, então $\text{tg } \alpha < 0$, pelo que $\overline{AC} = -\text{tg } \alpha$.

Logo, $A(\alpha) = \frac{\text{sen } \alpha}{2} + \left(-\frac{\text{tg } \alpha}{2}\right) = \frac{\text{sen } \alpha - \text{tg } \alpha}{2}$.

36.2. $\text{sen}(\pi - \alpha) + \cos\left(\frac{3\pi}{2} + \alpha\right) = \text{sen } \frac{\pi}{6}$

$\Leftrightarrow \text{sen } \alpha - \cos\left(\frac{\pi}{2} - \alpha\right) = \frac{1}{2}$

$\Leftrightarrow \text{sen } \alpha + \text{sen } \alpha = \frac{1}{2}$

$\Leftrightarrow 2 \text{ sen } \alpha = \frac{1}{2} \Leftrightarrow \text{sen } \alpha = \frac{1}{4}$

Da fórmula fundamental da trigonometria, vem que:

$\left(\frac{1}{4}\right)^2 + \cos^2\alpha = 1 \Leftrightarrow \cos^2\alpha = 1 - \frac{1}{16} \Leftrightarrow \cos^2\alpha = \frac{15}{16}$

Como $\alpha \in \left]\frac{\pi}{2}, \pi\right[$, então $\cos \alpha = -\sqrt{\frac{15}{16}} = -\frac{\sqrt{15}}{4}$.

Assim, $\text{tg } \alpha = \frac{\frac{1}{4}}{-\frac{\sqrt{15}}{4}} = -\frac{1}{\sqrt{15}} = -\frac{\sqrt{15}}{15}$.

Logo, $A(\alpha) = \frac{\frac{1}{4} - \left(-\frac{\sqrt{15}}{15}\right)}{2} = \frac{15 + 4\sqrt{15}}{120}$ u.a.

4. Funções trigonométricas e fenómenos periódicos

Funções seno, cosseno e tangente

Exercícios – páginas 15 e 16

37.

37.1. • $-1 \leq \text{sen } x \leq 1 \Leftrightarrow -3 \leq 3 \text{ sen } x \leq 3 \Leftrightarrow 2 \leq 5 + 3 \text{ sen } x \leq 8$

Logo, $D'_f = [2, 8]$.

• $-1 \leq \cos\left(\frac{1}{2}x\right) \leq 1 \Leftrightarrow -6 \leq 6 \cos\left(\frac{1}{2}x\right) \leq 6$

$\Leftrightarrow 6 \geq -6\cos\left(\frac{1}{2}x\right) \geq -6 \Leftrightarrow 9 \geq 3 - 6 \cos\left(\frac{1}{2}x\right) \geq -3$

Logo, $D'_g = [-3, 9]$.

• $-1 \leq \cos x \leq 1 \Leftrightarrow 2 \leq 3 + \cos x \leq 4$

$\Leftrightarrow \frac{1}{2} \geq \frac{1}{3 + \cos x} \geq \frac{1}{4} \Leftrightarrow 1 \geq \frac{1}{3 + \cos x} \geq \frac{1}{2}$

Logo, $D'_h = \left[\frac{1}{2}, 1\right]$.

• $D'_i = \mathbb{R}$

• $D'_j = \mathbb{R}$

• $0 \leq \cos^2\left(2x + \frac{\pi}{4}\right) \leq 1$

$\Leftrightarrow 0 \leq 4\cos^2\left(2x + \frac{\pi}{4}\right) \leq 4$

$\Leftrightarrow 0 \geq -4\cos^2\left(2x + \frac{\pi}{4}\right) \geq -4$

$\Leftrightarrow 3 \geq 3 - 4\cos^2\left(2x + \frac{\pi}{4}\right) \geq -1$

Logo, $D'_k = [-1, 3]$.

37.2. • Uma vez que $D'_f = [2, 8]$, conclui-se que a função f não tem zeros.

• Uma vez que $D'_g = [-3, 9]$, então $0 \in D'_g$ e conclui-se que a função g tem zeros.

• Uma vez que $D'_h = \left[\frac{1}{2}, 1\right]$, conclui-se que a função h não tem zeros.

• Uma vez que $D'_i = \mathbb{R}$, então $0 \in D'_i$ e conclui-se que a função i tem zeros.

• Uma vez que $D'_j = \mathbb{R}$, então $0 \in D'_j$ e conclui-se que a função j tem zeros.

• Uma vez que $D'_k = [-1, 3]$, então $0 \in D'_k$ e conclui-se que a função k tem zeros.

38.

38.1. $f(x) = \dfrac{\text{sen}\left(\dfrac{\pi}{2} + x\right)}{1 + \text{sen}(\pi + x)} + \dfrac{\cos(\pi - x)}{1 + \cos\left(\dfrac{\pi}{2} - x\right)} =$

$= \dfrac{\cos x}{1 - \text{sen}\, x} + \dfrac{-\cos x}{1 + \text{sen}\, x} =$

$= \dfrac{\cos x}{1 - \text{sen}\, x} \times \dfrac{1 + \text{sen}\, x}{1 + \text{sen}\, x} - \dfrac{\cos x}{1 + \text{sen}\, x} \times \dfrac{1 - \text{sen}\, x}{1 - \text{sen}\, x} =$

$= \dfrac{\cos x + \text{sen}\, x \cos x - \cos x + \text{sen}\, x \cos x}{1 - \text{sen}^2 x} =$

$= \dfrac{2\, \text{sen}\, x \cos x}{\cos^2 x} =$

$= 2\dfrac{\text{sen}\, x}{\cos x} =$

$= 2\, \text{tg}\, x$

38.2. $\text{sen}^2\theta + \cos^2\theta = 1 \Leftrightarrow \cos^2\theta = 1 - \dfrac{1}{16} \Leftrightarrow \cos^2\theta = \dfrac{15}{16}$.

Como $\theta \in \left]\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right[$, tem-se que $\cos\theta = -\dfrac{\sqrt{15}}{4}$.

$\text{tg}\,\theta = \dfrac{\text{sen}\,\theta}{\cos\theta} = \dfrac{-\dfrac{1}{4}}{-\dfrac{\sqrt{15}}{4}} = \dfrac{1}{\sqrt{15}} = \dfrac{\sqrt{15}}{15}$

Logo, $f(\theta) = 2\,\text{tg}\,\theta = \dfrac{2\sqrt{15}}{15}$.

39.

39.1. Para todo $x \in \mathbb{R}$, $-1 \leq \text{sen}\left(3x + \dfrac{\pi}{6}\right) \leq 1$

$\Leftrightarrow -3 \leq 3\,\text{sen}\left(3x + \dfrac{\pi}{6}\right) \leq 3$

$\Leftrightarrow 3 \geq -3\,\text{sen}\left(3x + \dfrac{\pi}{6}\right) \geq -3$

$\Leftrightarrow 8 \geq 5 - 3\,\text{sen}\left(3x + \dfrac{\pi}{6}\right) \geq 2$

Então, $D'_f = [2, 8]$.

39.2. $f\left(x - \dfrac{\pi}{2}\right) + f\left(x + \dfrac{\pi}{2}\right) =$

$= 5 - 3\,\text{sen}\left(3\left(x - \dfrac{\pi}{2}\right) + \dfrac{\pi}{6}\right) + 5 - 3\,\text{sen}\left(3\left(x + \dfrac{\pi}{2}\right) + \dfrac{\pi}{6}\right) =$

$= 10 - 3\,\text{sen}\left(3x - \dfrac{3\pi}{2} + \dfrac{\pi}{6}\right) - 3\,\text{sen}\left(3x + \dfrac{3\pi}{2} + \dfrac{\pi}{6}\right) =$

$= 10 - 3\,\text{sen}\left(3x + \dfrac{\pi}{2} + \dfrac{\pi}{6}\right) - 3\,\text{sen}\left(\pi + \left(3x + \dfrac{\pi}{2} + \dfrac{\pi}{6}\right)\right) =$

$= 10 - 3\,\text{sen}\left(3x + \dfrac{\pi}{2} + \dfrac{\pi}{6}\right) + 3\,\text{sen}\left(3x + \dfrac{\pi}{2} + \dfrac{\pi}{6}\right) =$

$= 10$

40.

40.1. $\dfrac{\cos^3 x}{1 - \text{sen}\, x} = \dfrac{\cos^2 x}{1 - \text{sen}\, x} \times \cos x =$

$= \dfrac{1 - \text{sen}^2 x}{1 - \text{sen}\, x} \times \cos x =$

$= \dfrac{(1 - \text{sen}\, x)(1 + \text{sen}\, x)}{1 - \text{sen}\, x} \times \cos x =$

$= (1 + \text{sen}\, x)\cos x =$

$= \cos x + \text{sen}\, x \cos x$

40.2. $\text{tg}(\pi - \alpha) = -2 \Leftrightarrow -\text{tg}\,\alpha = -2 \Leftrightarrow \text{tg}\,\alpha = 2$

$1 + \text{tg}^2\alpha = \dfrac{1}{\cos^2\alpha} \Leftrightarrow 1 + 2^2 = \dfrac{1}{\cos^2\alpha} \Leftrightarrow \cos^2\alpha = \dfrac{1}{5}$

Como $x \in \left[0, \dfrac{\pi}{2}\right[$, então $\cos\alpha = \sqrt{\dfrac{1}{5}} = \dfrac{\sqrt{5}}{5}$.

$\text{sen}^2\alpha + \cos^2\alpha = 1 \Leftrightarrow \text{sen}^2\alpha + \dfrac{1}{5} = 1 \Leftrightarrow \text{sen}^2\alpha = \dfrac{4}{5}$

Como $x \in \left[0, \dfrac{\pi}{2}\right[$, então $\text{sen}\, x = \sqrt{\dfrac{4}{5}} = \dfrac{2\sqrt{5}}{5}$.

Logo,

$f\left(\dfrac{\pi}{2} + \alpha\right) = \cos\left(\dfrac{\pi}{2} + \alpha\right) + \text{sen}\left(\dfrac{\pi}{2} + \alpha\right)\cos\left(\dfrac{\pi}{2} + \alpha\right) =$

$= -\text{sen}\,\alpha + \cos\alpha(-\text{sen}\,\alpha) =$

$= -\text{sen}\,\alpha - \text{sen}\,\alpha\cos\alpha =$

$= -\dfrac{2\sqrt{5}}{5} - \dfrac{2\sqrt{5}}{5} \times \dfrac{\sqrt{5}}{5} =$

$= -\dfrac{2\sqrt{5}}{5} - \dfrac{2}{5} =$

$= -\dfrac{2 + 2\sqrt{5}}{5}$

40.3. a) As coordenadas do ponto *A* são $(\cos\beta, \text{sen}\,\beta)$

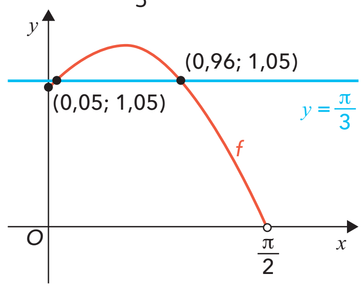
Assim, $\overline{AB} = 2\cos\beta$ e a altura do triângulo é $1 + \text{sen}\,\beta$.

Logo, a área do triângulo [*ABC*] é dada por

$\dfrac{2\cos\beta \times (1 + \text{sen}\,\beta)}{2} = \cos\beta + \text{sen}\,\beta\cos\beta = f(\beta)$.

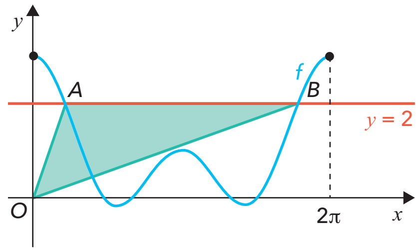
b) $A_{\text{Círculo}} = \pi \times 1^2 = \pi$

Pretende-se determinar os valores de β para os quais $f(\beta) = \dfrac{\pi}{3}$.



Assim, os valores pedidos são 0,05 e 0,96.

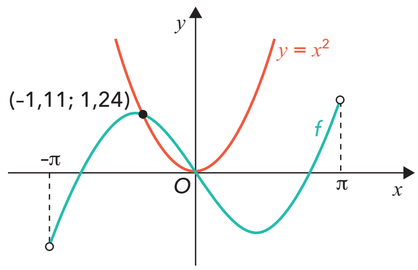
41.



$A(0{,}67; 2)$ e $B(5{,}61; 2)$

Logo, $A_{[OAB]} = \dfrac{\overline{AB} \times h}{2} \approx \dfrac{(5{,}61 - 0{,}67) \times 2}{2} \approx 4{,}9$ u.a.

42. Pretende-se resolver a equação $f(x) = x^2$.



As coordenadas do outro ponto nas condições do enunciado são $(-1{,}11; 1{,}24)$.

Fenómenos periódicos. Estudo de funções periódicas

Exercícios – páginas 17 e 18

43. Para todo $x \in \mathbb{R}$, $f(x + \pi) = 3 - \cos(2(x + \pi)) =$
$= 3 - \cos(2x + 2\pi) =$
$= 3 - \cos(2x) = f(x)$

Logo, f é uma função periódica de período π.

Para todo $x \in \mathbb{R}$, $g(x + 4\pi) =$

$$= 5 + 4 \text{ sen}\left(\frac{1}{2}(x + 4\pi) + 3\right) =$$

$$= 5 + 4 \text{ sen}\left(\frac{1}{2}x + 2\pi + 3\right) =$$

$$= 5 + 4 \text{ sen}\left(\frac{1}{2}x + 3\right) = g(x)$$

Logo, g é uma função periódica de período 4π.

Para todo $x \in D_h$, $h\left(x + \frac{\pi}{5}\right) =$

$$= 2 + \text{tg}\left(5\left(x + \frac{\pi}{5}\right) + \frac{\pi}{4}\right) =$$

$$= 2 + \text{tg}\left(5x + \pi + \frac{\pi}{4}\right) =$$

$$= 2 + \text{tg}\left(5x + \frac{\pi}{4}\right) = h(x)$$

Logo, h é uma função periódica de período $\frac{\pi}{5}$.

44. A função f tem período $\frac{2\pi}{\frac{1}{2}} = 4\pi$ e está representada no gráfico II.

A função g tem período $\frac{2\pi}{2} = \pi$ e está representada no gráfico IV.

A função h tem período 2π e está representada no gráfico I.

A função i tem período $\frac{2\pi}{\frac{1}{3}} = 6\pi$ e está representada no gráfico VI.

A função j tem período $\frac{2\pi}{\frac{1}{4}} = 8\pi$ e está representada no gráfico III.

A função k tem período $\frac{2\pi}{4} = \frac{\pi}{2}$ e está representada no gráfico V.

45.

45.1. $N(45 + t) = A + B \text{ sen}\left(\frac{2(45 + t)\pi}{45}\right) =$

$$= \text{sen}\left(\frac{90\pi + 2t\pi}{45}\right) =$$

$$= \text{sen}\left(2\pi + \frac{2t\pi}{45}\right) =$$

$$= \text{sen}\left(\frac{2t\pi}{45}\right) =$$

$= N(t)$, para todo $t \in [0, 52]$.

Logo, N é uma função periódica de período 45.

45.2. $N(0) = 10 \Leftrightarrow A + B \text{ sen } 0 = 10 \Leftrightarrow A = 10$

$$N(9) = 12{,}5 \Leftrightarrow 10 + B \text{ sen}\left(\frac{2 \times 9\pi}{45}\right) = 12{,}5$$

$$\Leftrightarrow B \text{ sen}\left(\frac{2\pi}{5}\right) = 2{,}5$$

$$\Leftrightarrow B = \frac{2{,}5}{\text{sen}\left(\frac{2\pi}{5}\right)}$$

Logo, $B \approx 2{,}63$.

45.3. $N(t) = 10 + 5 \text{ sen}\left(\frac{2t\pi}{45}\right)$

$$-1 \leq \text{sen}\left(\frac{2t\pi}{45}\right) \leq 1 \Leftrightarrow -5 \leq 5 \text{ sen}\left(\frac{2t\pi}{45}\right) \leq 5$$

$$\Leftrightarrow 5 \leq 10 + 5 \text{ sen}\left(\frac{2t\pi}{45}\right) \leq 15$$

O número mínimo de aves é 5000 e o número máximo é 15 000.

46. $A(t) = a + b \text{ sen}(ct + d)$

Para todo $t \in [0, 24[$,

$-1 \leq \text{sen}(ct + d) \leq 1$

$\Leftrightarrow -b \leq b \text{ sen}(ct + d) \leq b$

$\Leftrightarrow a - b \leq a - b \text{ sen}(ct + d) \leq a + b$

Uma vez que a temperatura mínima da água é 24,5 ºC e que a temperatura máxima da água é 25,5 ºC, tem-se:

$$\begin{cases} a - b = 24{,}5 \\ a + b = 25{,}5 \end{cases} \Leftrightarrow \begin{cases} a = 24{,}5 + b \\ 24{,}5 + b + b = 25{,}5 \end{cases} \Leftrightarrow \begin{cases} \text{———} \\ 2b = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} a = 25 \\ b = 0{,}5 \end{cases}$$

Logo, $A(t) = 25 + 0{,}5 \text{ sen}(ct + d)$.

Além disso, $A(0) = 25 \Leftrightarrow 25 + 0{,}5 \text{ sen}(0 + d) = 25$

$\Leftrightarrow 0{,}5 \text{ sen}(d) = 0 \Leftrightarrow \text{sen}(d) = 0$

Como $d \in \mathbb{Q}$, então $d = 0$.

Logo, $A(t) = 25 + 0{,}5 \text{ sen}(ct)$.

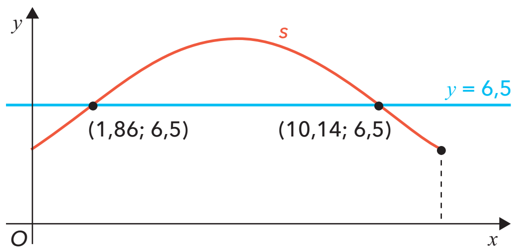
Tem-se ainda que o período da função é 10, ou seja,

$$\frac{2\pi}{c} = 10 \Leftrightarrow c = \frac{2\pi}{10} \Leftrightarrow c = \frac{\pi}{5}.$$

Então, $a = 25$, $b = 0{,}5$, $c = \frac{\pi}{5}$ e $d = 0$.

47.

47.1. Pretende-se resolver a equação $s(t) = 6{,}5$.



1,86 h corresponde a 1 hora e 52 minutos.

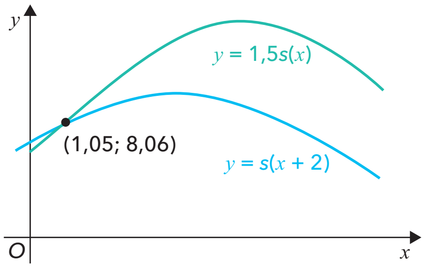
Cálculo auxiliar
$\frac{1}{60} = \frac{0{,}86}{x} \Leftrightarrow x = 60 \times 0{,}86 \Leftrightarrow x = 51{,}6$

10,14 h corresponde a 10 horas e 8 minutos.

Cálculo auxiliar
$\frac{1}{60} = \frac{0{,}14}{x} \Leftrightarrow x = 60 \times 0{,}14 \Leftrightarrow x = 8{,}4$

Assim, o número de pessoas no centro comercial foi 6500 às 11 horas e 52 minutos e às 20 horas e 8 minutos.

47.2. Pretende-se resolver a equação $s(t+2) = 1{,}5s(t)$.



1,05 h corresponde a 1 hora e 3 minutos.

> **Cálculo auxiliar**
> $\frac{1}{60} = \frac{0{,}05}{x} \Leftrightarrow x = 60 \times 0{,}05 \Leftrightarrow x = 3$

O momento referido aconteceu às 11 horas e 3 minutos.

Exercícios globais – páginas 19 a 25

48. $A_{[BCDE]} = 36$, logo $\overline{DE} = \overline{BE} = 6$

$\text{sen } 30° = \frac{\overline{BE}}{\overline{AD}} \Leftrightarrow \frac{1}{2} = \frac{6}{\overline{AD}} \Leftrightarrow \overline{AD} = 12$

$\text{tg } 30° = \frac{\overline{DE}}{\overline{AE}} \Leftrightarrow \frac{\sqrt{3}}{3} = \frac{6}{\overline{AE}} \Leftrightarrow \overline{AE} = \frac{8}{\sqrt{3}} \Leftrightarrow \overline{AE} = 6\sqrt{3}$

Assim, $P_{[ABCD]} = \overline{AE} + \overline{EB} + \overline{BC} + \overline{CD} + \overline{AD} =$

$= 6\sqrt{3} + 6 + 6 + 6 + 12 = (30 + 6\sqrt{3})$ u.c.

49. Opção (B)

$-2025° = -225° - 5 \times 360°$

Logo, o ângulo de amplitude $-2025°$ pertence ao segundo quadrante.

50. Opção (D)

$\frac{180}{\pi} = \frac{x}{1} \Leftrightarrow x = \frac{180}{\pi}$

Assim, $x \approx 57°$.

51. Opção (B)

Num intervalo de 35 minutos, a extremidade do ponteiro dos minutos percorre $\frac{7}{12}$ do total percorrido numa hora.

Ora, o total percorrido numa hora é $2 \times \pi \times 12 = 24\pi$.

Assim, em 35 minutos, a distância percorrida é $\frac{7}{12} \times 24\pi = 14\pi$.

52. Opção (C)

A opção (A) é falsa, uma vez que se $x \in \left]\frac{\pi}{2}, \pi\right[$ se tem $\text{sen } x > 0$ e $\cos x < 0$, ou seja, neste intervalo, há valores de x para os quais se tem $\text{sen } x \times \cos x < 0$.

A opção (B) é falsa, uma vez que se $x \in \left]\frac{\pi}{2}, \pi\right[$ se tem $\text{sen } x > 0$ e $\cos x < 0$, ou seja, neste intervalo, há valores de x para os quais se tem $\frac{\text{sen } x}{\cos x} < 0$.

A opção (C) é verdadeira, uma vez que se $x \in \left]\frac{3\pi}{2}, 2\pi\right[$ se tem $\text{sen } x < 0$ e $\cos x > 0$, ou seja, neste intervalo, há valores de x para os quais se tem $\text{sen } x \times \cos x < 0$.

A opção (D) é falsa, uma vez que se $x \in \left]0, \frac{\pi}{2}\right[$ se tem $\text{sen } x > 0$ e $\cos x > 0$, ou seja, neste intervalo, para qualquer valor de x, tem-se $\frac{\text{sen } x}{\cos x} > 0$.

53. Opção (A)

$\frac{1}{3 - \text{tg}^2\left(\frac{\pi}{4}\right)} = \frac{1}{3-1} = \frac{1}{2}$, logo $\frac{\pi}{4} \in D_f$ e a opção (A) é verdadeira.

$\frac{1}{3 - \text{tg}^2\left(\frac{\pi}{3}\right)} = \frac{1}{3 - (\sqrt{3})^2} = \frac{1}{3-3} = \frac{1}{0}$, logo $\frac{\pi}{3} \notin D_f$ e a opção (B) é falsa.

$\text{tg}\left(\frac{\pi}{2}\right)$ não existe, logo $\frac{\pi}{2} \notin D_f$ e a opção (C) é falsa.

$\frac{1}{3 - \text{tg}^2\left(\frac{2\pi}{3}\right)} = \frac{1}{3 - (-\sqrt{3})^2} = \frac{1}{3-3} = \frac{1}{0}$, logo $\frac{2\pi}{3} \notin D_f$ e a opção (D) é falsa.

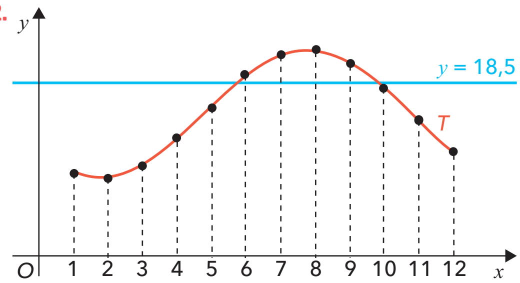
54. Opção (D)

$\frac{11\pi}{18} - \left(-\frac{\pi}{18}\right) = \frac{12\pi}{18} = \frac{2\pi}{3}$, que é o período positivo mínimo da função f.

55.

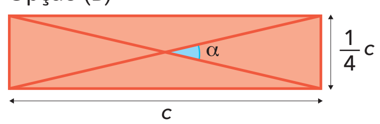
55.1. $T(6) = 6{,}9 \text{ sen}\left(\frac{\pi(6 - 4{,}8)}{6}\right) + 15{,}2 \approx 19{,}3$ °C

55.2.



A temperatura média é superior a 18,5 °C em junho, julho, agosto e setembro.

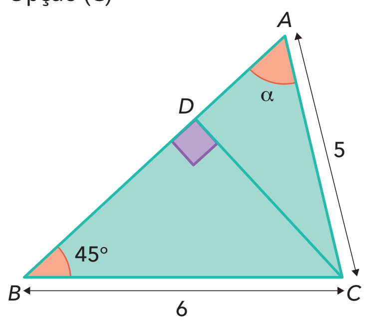
56. Opção (B)



$\text{tg}\left(\frac{\alpha}{2}\right) = \frac{\frac{1}{4}c}{\frac{2}{\frac{c}{2}}} \Leftrightarrow \text{tg}\left(\frac{\alpha}{2}\right) = \frac{\frac{1}{4}c}{c} \Leftrightarrow \text{tg}\left(\frac{\alpha}{2}\right) = \frac{1}{4}$

Logo, $\frac{\alpha}{2} \in]0°, 90°[$ e $\frac{\alpha}{2} = \text{tg}^{-1}\left(\frac{1}{4}\right)$, ou seja $\frac{\alpha}{2} \approx 14{,}04°$ e, portanto, $\alpha \approx 28°$.

57. Opção (C)



$\text{sen } 45^\circ = \frac{\overline{CD}}{\overline{BC}} \Leftrightarrow \frac{\sqrt{2}}{2} = \frac{\overline{CD}}{6} \Leftrightarrow \overline{CD} = 3\sqrt{2}$

$\text{sen } \alpha = \frac{\overline{CD}}{5} \Leftrightarrow \text{sen } \alpha = \frac{3\sqrt{2}}{5}$

Pela fórmula fundamental da trigonometria,

$\text{sen}^2\alpha + \cos^2\alpha = 1 \Leftrightarrow \left(\frac{3\sqrt{2}}{5}\right)^2 + \cos^2\alpha = 1$

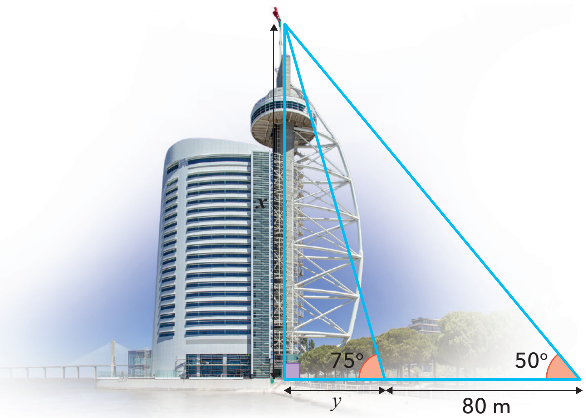
$\Leftrightarrow \cos^2\alpha = 1 - \frac{18}{25}$

$\Leftrightarrow \cos^2\alpha = \frac{7}{25}$

$\Leftrightarrow \cos\alpha = \pm\sqrt{\frac{7}{25}}$

Uma vez que α é um ângulo agudo, então $\cos\alpha = \frac{\sqrt{7}}{5}$.

58.



$\text{tg } 75^\circ = \frac{x}{y} \Leftrightarrow x = y \text{ tg } 75^\circ$

$\text{tg } 50^\circ = \frac{x}{y + 80} \Leftrightarrow (y + 80) \text{ tg } 50^\circ = x$

$\Leftrightarrow y \text{ tg } 50^\circ + 80 \text{ tg } 50^\circ = y \text{ tg } 75^\circ$

$\Leftrightarrow y \text{ tg } 75^\circ - y \text{ tg } 50^\circ = 80 \text{ tg } 50^\circ$

$\Leftrightarrow y(\text{tg } 75^\circ - \text{tg } 50^\circ) = 80 \text{ tg } 50^\circ$

$\Leftrightarrow y = \frac{80 \text{ tg } 50^\circ}{\text{tg } 75^\circ - \text{tg } 50^\circ}$

Logo, $x = \frac{80 \text{ tg } 50^\circ}{\text{tg } 75^\circ - \text{tg } 50^\circ} \times \text{tg } 75^\circ \approx 140$.

Assim, a altura da Torre Vasco da Gama é 140 metros.

59. Opção (D)

Para todo $x \in \left]\pi, \frac{3\pi}{2}\right[$, $-1 < \text{sen } x < 0$

$\Leftrightarrow -1 < \frac{5 - 3k}{2} < 0$

$\Leftrightarrow -2 < 5 - 3k < 0$

$\Leftrightarrow -7 < -3k < -5$

$\Leftrightarrow 7 > 3k > 5$

$\Leftrightarrow \frac{7}{3} > k > \frac{5}{3}$

Logo, $k \in \left]\frac{5}{3}, \frac{7}{3}\right[$.

60. Opção (A)

Uma vez que $\alpha \in \left]-2\pi, -\frac{3\pi}{2}\right[$, então $\text{sen } \alpha > 0$ e $\cos\alpha > 0$.

Uma vez que $\beta \in \left]\pi, \frac{3\pi}{2}\right[$, então $\text{sen } \beta < 0$ e $\cos\beta < 0$.

Assim, $\cos\alpha - \text{sen } \beta > 0$, ou seja, a opção (A) é verdadeira.

$\text{sen } \alpha \times \text{sen } \beta < 0$, pelo que a opção (B) é falsa.

$\cos\beta - \text{tg } \alpha < 0$, pelo que a opção (C) é falsa.

$\cos\alpha \times \cos\beta < 0$, pelo que a opção (D) é falsa.

61. Opção (C)

$\cos(\pi + \alpha) > 0 \Leftrightarrow -\cos\alpha > 0 \Leftrightarrow \cos\alpha < 0$

$\cos\left(\frac{\pi}{2} + \alpha\right) > 0 \Leftrightarrow -\text{sen } \alpha > 0 \Leftrightarrow \text{sen } \alpha < 0$

Assim:

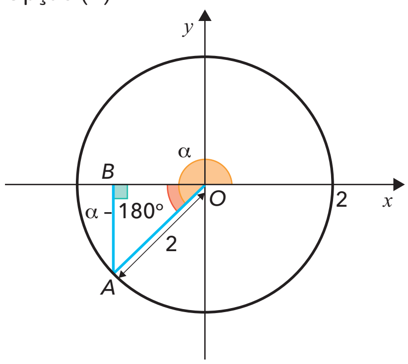
$\text{sen}(\pi + \alpha) = -\text{sen } \alpha > 0$, pelo que a opção (A) é falsa.

$\text{tg}(\alpha - \pi) = \text{tg } \alpha > 0$, pelo que a opção (B) é falsa.

$\text{tg}(\pi + \alpha) = \text{tg } \alpha > 0$, pelo que a opção (C) é verdadeira.

$\text{sen}\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha < 0$, pelo que a opção (D) é falsa.

62. Opção (D)



$\alpha - 180^\circ = 225^\circ - 180^\circ = 45^\circ$

$\cos 45^\circ = \frac{\overline{OB}}{2} \Leftrightarrow \frac{\sqrt{2}}{2} = \frac{\overline{OB}}{2} \Leftrightarrow \overline{OB} = \sqrt{2}$

Logo, a abcissa de A é $-\sqrt{2}$.

$\text{sen } 45^\circ = \frac{\overline{AB}}{2} \Leftrightarrow \frac{\sqrt{2}}{2} = \frac{\overline{AB}}{2} \Leftrightarrow \overline{AB} = \sqrt{2}$

Logo, a ordenada de A é $-\sqrt{2}$.

Assim, as coordenadas de A são $(-\sqrt{2}, -\sqrt{2})$.

63. Opção (A)

$\overline{AO} = \overline{OC} = 1$

$\overline{OD} = \cos\alpha$, logo $\overline{BD} = 1 + \cos\alpha$

$\overline{AD} = \text{sen } \alpha$

Pelo teorema de Pitágoras,

$\overline{AB}^2 = (1 + \cos\alpha)^2 + \text{sen}^2\alpha$

$\Leftrightarrow \overline{AB}^2 = 1 + 2\cos\alpha + \cos^2\alpha + \text{sen}^2\alpha$

$\Leftrightarrow \overline{AB}^2 = 1 + 2\cos\alpha + 1$

$\Leftrightarrow \overline{AB}^2 = 2 + 2\cos\alpha$

Logo, $\overline{AB} = \sqrt{2 + 2\cos\alpha}$.

Assim, $P_{[ABCO]} = 2 + 2\sqrt{2 + 2\cos\alpha}$.

Tem-se então:

$2 + 2\sqrt{2 + 2\cos\alpha} = 2 + 2\sqrt{3}$

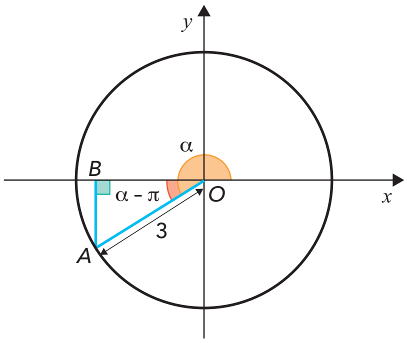
$\Leftrightarrow \sqrt{2 + 2\cos\alpha} = \sqrt{3}$

$\Leftrightarrow 2 + 2\cos\alpha = 3$

$\Leftrightarrow \cos\alpha = \frac{1}{2}$

Como $\alpha \in \left]0, \frac{\pi}{2}\right[$, então $\alpha = \frac{\pi}{3}$.

64.



Como a ordenada do ponto A é $-\frac{9}{5}$, então

$\overline{AB} = \frac{9}{5}$

Assim, $\text{sen}\,(\alpha - \pi) = \frac{\overline{AB}}{3} \Leftrightarrow -\text{sen}\,\alpha = \frac{\frac{9}{5}}{3} \Leftrightarrow \text{sen}\,\alpha = -\frac{3}{5}$.

Pela fórmula fundamental da trigonometria,

$$\text{sen}^2\alpha + \cos^2\alpha = 1 \Leftrightarrow \left(-\frac{3}{5}\right)^2 + \cos^2\alpha = 1$$
$$\Leftrightarrow \cos^2\alpha = 1 - \frac{9}{25}$$
$$\Leftrightarrow \cos^2\alpha = \frac{16}{25}$$
$$\Leftrightarrow \cos\alpha = \pm\frac{4}{5}$$

Uma vez que α pertence ao terceiro quadrante, então $\cos\alpha = -\frac{4}{5}$.

Assim:

$$\cos\left(\frac{\pi}{2} + \alpha\right) - \text{sen}\left(\frac{3\pi}{2} - \alpha\right) + \text{tg}(2025\pi - \alpha) =$$
$$= -\text{sen}\,\alpha - \text{sen}\left(\pi + \frac{\pi}{2} - \alpha\right) + \text{tg}(\pi - \alpha) =$$
$$= -\text{sen}\,\alpha + \text{sen}\left(\frac{\pi}{2} - \alpha\right) - \text{tg}\,\alpha =$$
$$= -\text{sen}\,\alpha + \cos\alpha - \frac{\text{sen}\,\alpha}{\cos\alpha} =$$
$$= -\left(-\frac{3}{5}\right) + \left(-\frac{4}{5}\right) - \frac{-\frac{3}{5}}{-\frac{4}{5}} =$$
$$= \frac{3}{5} - \frac{4}{5} - \frac{3}{4} =$$
$$= -\frac{19}{20}$$

65. Opção (A)

Seja M o ponto médio de $[AB]$.

As coordenadas do ponto A são $(\cos\alpha, \text{sen}\,\alpha)$.

Tem-se assim que $\overline{OM} = \text{sen}\,\alpha$ e $\overline{AM} = \overline{MB} = \overline{OC} = \cos\alpha$.

$$\text{Assim, } A_{[ABCO]} = \frac{(2\cos\alpha + \cos\alpha) \times \text{sen}\,\alpha}{2} =$$
$$= \frac{3\cos\alpha\,\text{sen}\,\alpha}{2} = \frac{3}{2}\text{sen}\,\alpha\cos\alpha.$$

66.

$$\cos\alpha = \frac{\overline{BD}}{\overline{DE}} \Leftrightarrow \cos\alpha = \frac{\overline{BD}}{8} \Leftrightarrow \overline{BD} = 8\cos\alpha$$
$$\text{sen}\,\alpha = \frac{\overline{BE}}{\overline{DE}} \Leftrightarrow \text{sen}\,\alpha = \frac{\overline{BE}}{8} \Leftrightarrow \overline{BE} = 8\,\text{sen}\,\alpha$$

Logo, $\overline{BC} = 6 + 8\,\text{sen}\,\alpha$

$$\text{tg}(2\alpha) = \frac{\overline{BC}}{\overline{AB}} \Leftrightarrow \text{tg}(2\alpha) = \frac{6 + 8\,\text{sen}\,\alpha}{\overline{AB}}$$
$$\Leftrightarrow \overline{AB} = \frac{6 + 8\,\text{sen}\,\alpha}{\text{tg}(2\alpha)}$$

Logo, $\overline{AD} = \overline{AB} + \overline{BD} = 8\cos\alpha + \frac{6 + 8\,\text{sen}\,\alpha}{\text{tg}(2\alpha)}$.

67.

67.1. a) $\text{tg}\,x = \frac{1}{\overline{EF}} \Leftrightarrow \overline{EF} = \frac{1}{\text{tg}\,x} \Leftrightarrow \overline{AB} = \frac{1}{\text{tg}\,x}$

$$\overline{AB} = \frac{3}{4}\overline{AD} \Leftrightarrow \frac{1}{\text{tg}\,x} = \frac{3}{4}\overline{AD} \Leftrightarrow \overline{AD} = \frac{4}{3\,\text{tg}\,x}$$

Logo, o volume do paralelepípedo é dado por:

$$V = \frac{1}{\text{tg}\,x} \times \frac{4}{3\,\text{tg}\,x} \times 1 = \frac{4}{3\,\text{tg}^2 x}.$$

b) Da alínea anterior, vem que $\overline{AB} = \frac{1}{\text{tg}\,x}$ e $\overline{AD} = \frac{4}{3\,\text{tg}\,x}$

Assim, área total do paralelepípedo é dada por:

$$A = 2 \times \frac{1}{\text{tg}\,x} \times \frac{4}{3\,\text{tg}\,x} + 2 \times \frac{1}{\text{tg}\,x} \times 1 + 2 \times \frac{4}{3\,\text{tg}\,x} \times 1 =$$
$$= \frac{8}{3\,\text{tg}^2 x} + \frac{2}{\text{tg}\,x} + \frac{8}{3\,\text{tg}\,x} = \frac{8}{3\,\text{tg}^2 x} + \frac{14}{3\,\text{tg}\,x}$$

67.2. Se $x = 30^\circ$, então:

$$V = \frac{4}{3\,\text{tg}^2 30^\circ} = \frac{4}{3 \times \left(\frac{\sqrt{3}}{3}\right)^2} = \frac{4}{3 \times \frac{1}{3}} = 4 \text{ u.v.}$$

$$\text{e } A = \frac{8}{3\,\text{tg}^2 30^\circ} + \frac{14}{3\,\text{tg}\,30^\circ} = \frac{8}{3 \times \left(\frac{\sqrt{3}}{3}\right)^2} + \frac{14}{3 \times \frac{\sqrt{3}}{3}} =$$
$$= \frac{8}{3 \times \frac{1}{3}} + \frac{14}{\sqrt{3}} = \left(8 + \frac{14\sqrt{3}}{3}\right) \text{ u.a.}$$

68.

68.1. $A_{[AEF]} = A_{[ABCD]} - A_{[ABE]} - A_{[AFD]} - A_{[ECF]}$

$A_{[ABCD]} = 1 \times 1 = 1$

$\text{tg}\,x = \frac{\overline{BE}}{1} \Leftrightarrow \overline{BE} = \text{tg}\,x$

Nas condições do enunciado, tem-se que $\overline{BE} = \overline{DF}$, logo $\overline{DF} = \text{tg}\,x$.

Além disso, $\overline{EC} = \overline{CF} = 1 - \text{tg}\,x$.

Logo, $A_{[ABE]} = A_{[AFD]} = \frac{1 \times \text{tg}\,x}{2} = \frac{\text{tg}\,x}{2}$ e

$A_{[ECF]} = \frac{(1 - \text{tg}\,x) \times (1 - \text{tg}\,x)}{2} = \frac{1 - 2\,\text{tg}\,x + \text{tg}^2 x}{2}$.

$$\text{Assim, } A_{[AEF]} = 1 - \frac{\text{tg}\,x}{2} - \frac{\text{tg}\,x}{2} - \frac{1 - 2\,\text{tg}\,x + \text{tg}^2 x}{2} =$$
$$= 1 - \text{tg}\,x - \frac{1}{2} + \text{tg}\,x - \frac{\text{tg}^2 x}{2} =$$
$$= \frac{1}{2} - \frac{\text{tg}^2 x}{2} =$$
$$= \frac{1}{2} - \frac{\frac{\text{sen}^2 x}{\cos^2 x}}{2} =$$
$$= \frac{1}{2} - \frac{\text{sen}^2 x}{2\cos^2 x} =$$
$$= \frac{\cos^2 x - (1 - \cos^2 x)}{2\cos^2 x} =$$
$$= \frac{2\cos^2 x - 1}{2\cos^2 x} =$$
$$= 1 - \frac{1}{2\cos^2 x} = A(x)$$

68.2. $A(x) = \frac{1}{2}$

$\Leftrightarrow 1 - \frac{1}{2\cos^2 x} = \frac{1}{2}$

$\Leftrightarrow \frac{1}{2\cos^2 x} = \frac{1}{2}$

$\Leftrightarrow \cos^2 x = 1$

$\Leftrightarrow \cos x = 1 \vee \cos x = -1$

Uma vez que $x \in \left[0, \frac{\pi}{4}\right[$, tem-se que $x = 0$.

68.3. $\text{tg}\left(\frac{\pi}{2} - x\right) = \frac{3}{2} \Leftrightarrow \frac{\text{sen}\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)} = \frac{3}{2} \Leftrightarrow \frac{\cos x}{\text{sen}\, x} = \frac{3}{2}$

$\Leftrightarrow \frac{1}{\text{tg}\, x} = \frac{3}{2} \Leftrightarrow \text{tg}\, x = \frac{2}{3}$

Então,

$1 + \text{tg}^2 x = \frac{1}{\cos^2 x} \Leftrightarrow 1 + \frac{4}{9} = \frac{1}{\cos^2 x}$

$\Leftrightarrow \frac{13}{9} = \frac{1}{\cos^2 x}$

$\Leftrightarrow \cos^2 x = \frac{9}{13}$

Logo, $A(x) = 1 - \frac{1}{2 \times \frac{9}{13}} = \frac{5}{18}$ u.a.

69.

69.1. $\overline{OC} = 1$

$P(\cos\alpha, \text{sen}\,\alpha)$

Seja D a projeção ortogonal de P sobre o eixo Ox.

$\overline{OD} = \cos\alpha$

Logo, $A = \frac{\cos\alpha \times 1}{2} = \frac{\cos\alpha}{2} = f(\alpha)$.

69.2. $\text{sen}\left(\frac{3\pi}{2} + \theta\right) - \cos(-3\pi + \theta) + \text{tg}(-\pi - \theta) = -\frac{3}{7}$

$\Leftrightarrow -\text{sen}\left(\frac{\pi}{2} + \theta\right) - \cos(\pi + \theta) - \text{tg}(\pi + \theta) = -\frac{3}{7}$

$\Leftrightarrow -\cos\theta + \cos\theta - \text{tg}\,\theta = -\frac{3}{7}$

$\Leftrightarrow \text{tg}\,\theta = \frac{3}{7}$

Assim, $1 + \text{tg}^2\theta = \frac{1}{\cos^2\theta}$

$\Leftrightarrow 1 + \frac{9}{49} = \frac{1}{\cos^2\theta}$

$\Leftrightarrow \frac{58}{49} = \frac{1}{\cos^2\theta}$

$\Leftrightarrow \cos^2\theta = \frac{49}{58}$

Logo, como $\theta \in \left]0, \frac{\pi}{2}\right[$, $\cos\theta = \sqrt{\frac{49}{58}} = \frac{7\sqrt{58}}{58}$.

Então, $f(\theta) = \frac{\cos\theta}{2} = \frac{7\sqrt{58}}{116}$.

69.3. $(1 - \text{sen}\,x)\left(\frac{1}{\cos x} + \text{tg}\,x\right) =$

$= (1 - \text{sen}\,x)\left(\frac{1}{\cos x} + \frac{\text{sen}\,x}{\cos x}\right) =$

$= (1 - \text{sen}\,x)\left(\frac{1 + \text{sen}\,x}{\cos x}\right) =$

$= \frac{1 - \text{sen}^2 x}{\cos x} =$

$= \frac{\cos^2 x}{\cos x} =$

$= \cos x =$

$= 2 \times \frac{\cos x}{2} = 2f(x)$

69.4. $P(\cos\alpha, \text{sen}\,\alpha)$

$C(0, -1)$

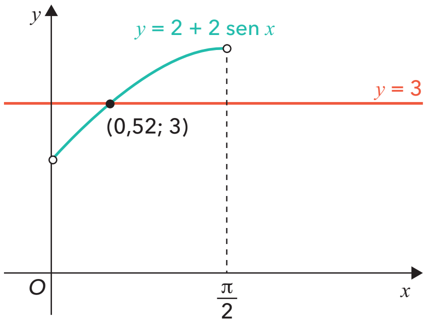
$\overline{CP} = \sqrt{\cos^2\alpha + (\text{sen}\,\alpha + 1)^2} =$

$= \sqrt{\cos^2\alpha + \text{sen}^2\alpha + 2\,\text{sen}\,\alpha + 1} =$

$= \sqrt{1 + 2\,\text{sen}\,\alpha + 1} =$

$= \sqrt{2 + 2\,\text{sen}\,\alpha}$

69.5. $A = 3 \Leftrightarrow \overline{CP}^2 = 3 \Leftrightarrow 2 + 2\,\text{sen}\,\alpha = 3$



Logo, $\alpha \approx 0{,}52$.

70.

70.1. $B(\cos\alpha, \sin\alpha)$ sendo $\cos\alpha < 0$ e $\sin\alpha < 0$, porque o ângulo pertence ao terceiro quadrante.

Seja M o ponto médio de $[BC]$.

$\overline{BM} = -\cos\alpha$

$\overline{BC} = -2\cos\alpha$

$\overline{OM} = -\text{sen}\,\alpha$

$\overline{AM} = 1 - \text{sen}\,\alpha$

Então, $A_{[ABC]} = \frac{\overline{BC} \times \overline{AM}}{2} =$

$= \frac{-2\cos\alpha \times (1 - \text{sen}\,\alpha)}{2} =$

$= -\cos\alpha + \cos\alpha\,\text{sen}\,\alpha =$

$= \text{sen}\,\alpha\cos\alpha - \cos\alpha = A(\alpha)$

70.2. $5\,\text{sen}(\pi - \alpha) = -\sqrt{5} \Leftrightarrow \text{sen}(\pi - \alpha) = -\frac{\sqrt{5}}{5}$

$\Leftrightarrow \text{sen}\,\alpha = -\frac{\sqrt{5}}{5}$

Pela fórmula fundamental da trigonometria,

$\text{sen}^2\alpha + \cos^2\alpha = 1 \Leftrightarrow \left(-\frac{\sqrt{5}}{5}\right)^2 + \cos^2\alpha = 1$

$\Leftrightarrow \cos^2\alpha = 1 - \frac{1}{5}$

$\Leftrightarrow \cos^2\alpha = \frac{4}{5}$

$\Leftrightarrow \cos\alpha = \pm\frac{2}{\sqrt{5}}$

$\Leftrightarrow \cos\alpha = \pm\frac{2\sqrt{5}}{5}$

Como o ângulo pertence ao terceiro quadrante, então $\cos\alpha = -\frac{2\sqrt{5}}{5}$.

Assim, $A(\alpha) = -\frac{\sqrt{5}}{5} \times \left(-\frac{2\sqrt{5}}{5}\right) - \left(-\frac{2\sqrt{5}}{5}\right) =$

$= \frac{2}{5} + \frac{2\sqrt{5}}{5} = \frac{2 + 2\sqrt{5}}{5}$ u.a.

71. $f(x) = \dfrac{\text{sen}^2 x}{\cos^2 x + 3\cos x + 2} =$

$= \dfrac{1 - \cos^2 x}{(\cos x + 1)(\cos x + 2)} =$

$= \dfrac{(1 + \cos x)\,(1 - \cos x)}{(1 + \cos x)(2 + \cos x)} =$

$= \dfrac{1 - \cos x}{2 + \cos x}$

> **Cálculo auxiliar**
>
> $\cos^2 x + 3\cos x + 2 = 0 \Leftrightarrow \cos x = \dfrac{-3 \pm \sqrt{9 - 8}}{2}$
>
> $\Leftrightarrow \cos x = -1 \ \vee \ \cos x = -2$

72.

72.1. $f(x) = \dfrac{1}{\text{tg}\, x + \dfrac{\cos x}{1 + \text{sen}\, x}} =$

$= \dfrac{1}{\dfrac{\text{sen}\, x}{\cos x} + \dfrac{\cos x}{1 + \text{sen}\, x}} =$

$= \dfrac{1}{\dfrac{\text{sen}\, x + \text{sen}^2 x + \cos^2 x}{\cos x\,(1 + \text{sen}\, x)}} =$

$= \dfrac{\cos x\,(1 + \text{sen}\, x)}{\text{sen}\, x + 1} =$

$= \cos x$

72.2. $\cos\left(\dfrac{7\pi}{2} - \alpha\right) = \dfrac{3}{5} \Leftrightarrow \cos\left(\dfrac{3\pi}{2} - \alpha\right) = \dfrac{3}{5}$

$\Leftrightarrow -\cos\left(\dfrac{\pi}{2} - \alpha\right) = \dfrac{3}{5}$

$\Leftrightarrow -\text{sen}\, \alpha = \dfrac{3}{5}$

$\Leftrightarrow \text{sen}\, \alpha = -\dfrac{3}{5}$

Assim, pela fórmula fundamental da trigonometria,

$\left(-\dfrac{3}{5}\right)^2 + \cos^2\alpha = 1 \Leftrightarrow \cos^2\alpha = 1 - \dfrac{9}{25}$

$\Leftrightarrow \cos^2\alpha = \dfrac{16}{25}$

Como $\alpha \in \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[$, então $\cos\alpha > 0$.

Logo,

$\cos\alpha = \sqrt{\dfrac{16}{25}} = \dfrac{4}{5}$.

Ou seja, $f(\alpha) = \dfrac{4}{5}$.

72.3. $f(x) = 1 + \text{sen}^2 x$

$\Leftrightarrow \cos x = 1 + \text{sen}^2 x$

$\Leftrightarrow \cos x = 1 + 1 - \cos^2 x$

$\Leftrightarrow \cos^2 x + \cos x - 2 = 0$

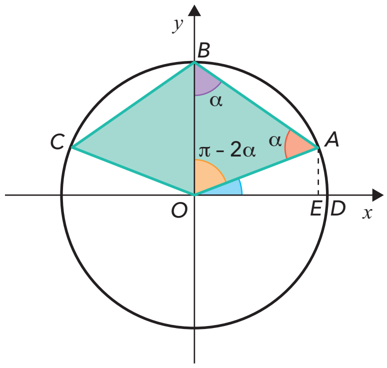
$\Leftrightarrow \cos x = \dfrac{-1 \pm \sqrt{1 + 8}}{2}$

$\Leftrightarrow \underbrace{\cos x = -2}_{\text{Equação impossível}} \ \vee \ \cos x = 1$

Como $x \in \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[$, então $x = 0$.

73.

73.1.



$\overline{OA} = \overline{OB} = 2$

Logo, $O\hat{B}A = B\hat{A}O = \alpha$ e $A\hat{O}B = \pi - 2\alpha$.

Seja D o ponto de interseção da circunferência com o semieixo positivo Ox.

$A\hat{O}D = \dfrac{\pi}{2} - (\pi - 2\alpha) = -\dfrac{\pi}{2} + 2\alpha$

Seja E a projeção ortogonal de A sobre Ox.

Então, $\cos\left(-\dfrac{\pi}{2} + 2\alpha\right) = \dfrac{\overline{OE}}{\overline{OA}}$

$\Leftrightarrow \cos\left(\dfrac{\pi}{2} - 2\alpha\right) = \dfrac{\overline{OE}}{2}$

$\Leftrightarrow \text{sen}(2\alpha) = \dfrac{\overline{OE}}{2}$

$\Leftrightarrow \overline{OE} = 2\,\text{sen}(2\alpha)$

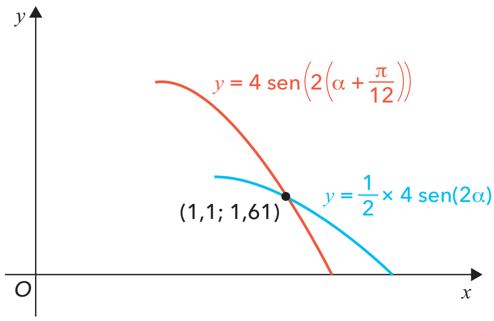
Logo, $A_{[OABC]} = 2 \times \dfrac{\overline{OB} \times \overline{OE}}{2} =$

$= 2 \times 2\,\text{sen}(2\alpha) =$

$= 4\,\text{sen}(2\alpha) = A(\alpha)$

73.2. Pretende-se resolver a equação $A\left(\alpha + \dfrac{\pi}{12}\right) = \dfrac{1}{2}A(\alpha)$

$\Leftrightarrow 4\,\text{sen}\left(2\left(\alpha + \dfrac{\pi}{12}\right)\right) = \dfrac{1}{2} \times 4\,\text{sen}(2\alpha)$



Logo, $\alpha \approx 1{,}1$.

74. $\dfrac{C_1}{C_2} = \dfrac{1}{2} \Leftrightarrow C_2 = 2C_1$

$C_1 + C_2 = 2\pi r \Leftrightarrow C_1 + 2C_1 = 2\pi r$

$\Leftrightarrow 3C_1 = 2\pi r \Leftrightarrow C_1 = \dfrac{2\pi}{3} r$

Logo, $A\hat{O}B = \dfrac{2\pi}{3}$.

Seja M o ponto médio de $[AB]$.

Se $A\hat{O}B = \dfrac{2\pi}{3}$, então $A\hat{O}M = \dfrac{\pi}{3}$.

Logo,

$\cos A\hat{O}M = \dfrac{\overline{OM}}{\overline{OA}} \Leftrightarrow \cos\dfrac{\pi}{3} = \dfrac{\overline{OM}}{r}$

$\Leftrightarrow \overline{OM} = r\cos\dfrac{\pi}{3}$

$\Leftrightarrow \overline{OM} = \dfrac{1}{2} r$

$\text{sen}\, A\hat{O}M = \dfrac{\overline{AM}}{\overline{OA}} \Leftrightarrow \text{sen}\, \dfrac{\pi}{3} = \dfrac{\overline{AM}}{r}$

$\Leftrightarrow \overline{AM} = r\, \text{sen}\, \dfrac{\pi}{3}$

$\Leftrightarrow \overline{AM} = \dfrac{\sqrt{3}}{2} r$

$S_1 = A_{\text{Setor circular}} - A_{[AOB]} =$

$= \dfrac{\dfrac{2\pi}{3} \times r^2}{2} - 2 \times \dfrac{\overline{AM} \times \overline{OM}}{2} =$

$= \dfrac{\pi}{3} r^2 - \dfrac{\sqrt{3}}{2} r \times \dfrac{1}{2} r =$

$= \dfrac{\pi}{3} r^2 - \dfrac{\sqrt{3}}{4} r^2 =$

$= r^2\left(\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{4}\right)$

$S_2 = A_{\text{Círculo}} - S_1 =$

$= \pi r^2 - r^2\left(\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{4}\right) =$

$= r^2\left(\dfrac{2\pi}{3} - \dfrac{\sqrt{3}}{4}\right)$

Logo, $\dfrac{S_1}{S_2} = \dfrac{r^2\left(\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{4}\right)}{r^2\left(\dfrac{2\pi}{3} - \dfrac{\sqrt{3}}{4}\right)} = \dfrac{\dfrac{4\pi - 3\sqrt{3}}{12}}{\dfrac{8\pi - 3\sqrt{3}}{12}} = \dfrac{4\pi - 3\sqrt{3}}{8\pi - 3\sqrt{3}}$.