PROPOSTAS DE RESOLUÇÃO

do Manual

1. Números complexos

Recorda – páginas 6 e 7

1.

a) $x^2 - 4x + 3 = 0 \Leftrightarrow x = \dfrac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 3}}{2}$

$\Leftrightarrow x = \dfrac{4 \pm \sqrt{4}}{2}$

$\Leftrightarrow x = \dfrac{4 \pm 2}{2}$

$\Leftrightarrow x = 3 \vee x = 1$

C.S. = {1, 3}

b) $x^2 - 4x + 4 = 0 \Leftrightarrow x = \dfrac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 4}}{2}$

$\Leftrightarrow x = \dfrac{4 \pm \sqrt{0}}{2}$

$\Leftrightarrow x = \dfrac{4}{2}$

$\Leftrightarrow x = 2$

C.S. = {2}

c) $x^2 + 2x = -5 \Leftrightarrow x^2 + 2x + 5 = 0$

$\Leftrightarrow x = \dfrac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 5}}{2}$

$\Leftrightarrow x = \dfrac{-2 \pm \sqrt{-16}}{2}$

Equação impossível em $\mathbb{R}$.

C.S. = $\varnothing$

2.

a) $x^2 - 5x + 6 = 0$

$\triangle = (-5)^2 - 4 \times 1 \times 6 = 25 - 24 = 1 > 0$

Logo, a equação $x^2 - 5x + 6 = 0$ tem duas soluções reais distintas.

b) $x^2 + 2x + 1 = 0$

$\triangle = 2^2 - 4 \times 1 \times 1 = 4 - 4 = 0$

Logo, a equação $x^2 + 2x + 1 = 0$ tem uma solução real.

c) $x^2 + 4x = -13 \Longleftrightarrow x^2 + 4x + 13 = 0$

$\triangle = 4^2 - 4 \times 1 \times 13 = 16 - 52 = -36 < 0$

Logo, a equação $x^2 + 4x = -13$ não tem soluções reais.

Capítulo 1 – Números complexos na forma algébrica

Tarefa – Novos números – página 8

Sabemos que $x + y = 6$ e $x \times y = 13$.

Assim:

a) **i.** $(x - 2)(y - 2) = xy - 2x - 2y + 4 = 13 - 2(x + y) + 4 = 13 - 2 \times 6 + 4 = 5$

ii. $x^2 + y^2 = (x + y)^2 - 2xy = 6^2 - 2 \times 13 = 10$

iii. $(x + 3)(y + 3) = xy + 3x + 3y + 9 = 13 + 3(x + y) + 9 = 13 + 3 \times 6 + 9 = 40$

b) Sabemos que $x + y = 6$ e $x \times y = 13$:

$\begin{cases} x + y = 6 \\ x \times y = 13 \end{cases} \Leftrightarrow \begin{cases} y = 6 - x \\ x \times (6 - x) = 13 \end{cases} \Leftrightarrow \begin{cases} y = 6 - x \\ 6x - x^2 = 13 \end{cases}$

$\Leftrightarrow \begin{cases} y = 6 - x \\ x^2 - 6x + 13 = 0 \end{cases}$

Resolvendo a equação de 2º grau $x^2 - 6x + 13 = 0$, obtém-se:

$x^2 - 6x + 13 = 0 \Leftrightarrow x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 1 \times 13}}{2 \times 1}$

$\Leftrightarrow x = \dfrac{6 \pm \sqrt{-16}}{2}$

Equação impossível no conjunto dos números reais, $\mathbb{R}$.

Portanto, concluiu-se que não existem números reais que satisfazem as condições do problema.

c) Investigação ao cuidado do aluno.

d) Voltando à resolução da equação $x^2 - 6x + 13 = 0$, e considerando agora que $\sqrt{-1} = i$, tem-se que:

$x^2 - 6x + 13 = 0 \Leftrightarrow x = \dfrac{6 \pm \sqrt{-16}}{2}$

$\Leftrightarrow x = \dfrac{6 \pm \sqrt{-1} \times \sqrt{16}}{2}$

$\Leftrightarrow x = \dfrac{6 \pm i \times 4}{2}$

$\Leftrightarrow x = 3 + 2i \vee x = 3 - 2i$

Assim, se $x = 3 + 2i$, vem que $y = 6 - x = 6 - (3 + 2i) = 3 - 2i$.

Se $x = 3 - 2i$, vem que $y = 6 - x = 6 - (3 - 2i) = 3 + 2i$.

e) Aplicando as operações com números reais a estes dois números novos, e considerando que $\sqrt{-1} = i$ (e consequentemente, $i^2 = -1$), verifica-se que a soma destes dois números é 6 e o seu produto é 13:

$x + y = 3 + 2i + 3 - 2i = 6$ e $x \times y = (3 + 2i) \times (3 - 2i) = 3^2 - (2i)^2 = 9 - 4i^2 = 9 - 4 \times (-1) = 9 + 4 = 13$

Exercícios de margem – páginas 11 a 27

1.

a) $x^2 - 4x + 5 = 0 \Leftrightarrow x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 1 \times 5}}{2 \times 1}$

$\Leftrightarrow x = \dfrac{4 \pm \sqrt{-4}}{2}$

$\Leftrightarrow x = \dfrac{4 \pm 2i}{2}$

$\Leftrightarrow x = 2 + i \vee x = 2 - i$

C.S. = {2 + i, 2 − i}

b) $2x^2 + 3x + 4 = 0 \Leftrightarrow x = \dfrac{-3 \pm \sqrt{3^2 - 4 \times 2 \times 4}}{2 \times 2}$

$\Leftrightarrow x = \dfrac{-3 \pm \sqrt{-23}}{4}$

$\Leftrightarrow x = \dfrac{-3 \pm \sqrt{23}i}{4}$

$\Leftrightarrow x = -\dfrac{3}{4} + \dfrac{\sqrt{23}}{4}i \vee x = -\dfrac{3}{4} - \dfrac{\sqrt{23}}{4}i$

C.S. = $\left\{-\dfrac{3}{4} + \dfrac{\sqrt{23}}{4}i, -\dfrac{3}{4} - \dfrac{\sqrt{23}}{4}i\right\}$

c) $x^3 + 16x = 0 \Leftrightarrow x(x^2 + 16) = 0$

$\Leftrightarrow x = 0 \vee x^2 = -16$

$\Leftrightarrow x = 0 \vee x = \pm\sqrt{-16}$

$\Leftrightarrow x = 0 \vee x = \pm 4i$

$\Leftrightarrow x = 0 \vee x = 4i \vee x = -4i$

C.S. = {0, 4i, − 4i}

d) $3x^3 + 2x^2 + x = 0$
$\Leftrightarrow x(3x^2 + 2x + 1) = 0$
$\Leftrightarrow x = 0 \vee 3x^2 + 2x + 1 = 0$
$\Leftrightarrow x = 0 \vee x = \frac{-2 \pm \sqrt{2^2 - 4 \times 3 \times 1}}{2 \times 3}$
$\Leftrightarrow x = 0 \vee x = \frac{-2 \pm \sqrt{-8}}{6}$
$\Leftrightarrow x = 0 \vee x = \frac{-2 \pm 2\sqrt{2}i}{6}$
$\Leftrightarrow x = 0 \vee x = -\frac{1}{3} + \frac{\sqrt{2}}{3}i \vee x = -\frac{1}{3} - \frac{\sqrt{2}}{3}i$
C.S. $= \left\{0, -\frac{1}{3} + \frac{\sqrt{2}}{3}i, -\frac{1}{3} - \frac{\sqrt{2}}{3}i\right\}$

2. Para que $k^2 - 1 + (p+1)i = 2 - 7i$, tem de se verificar $k^2 - 1 = 2 \wedge p + 1 = -7$. Assim:
$k^2 = 3 \wedge p = -8 \Leftrightarrow (k = \sqrt{3} \vee k = -\sqrt{3}) \wedge p = -8$

3.

a) $\text{Re}(z) = -2$ e $\text{Im}(z) = 3$

b) $\text{Re}(z) = 1$ e $\text{Im}(z) = -1$

c) $\text{Re}(z) = 0$ e $\text{Im}(z) = 2027$

d) $z = 5i^2 = 5 \times (-1) = -5$
$\text{Re}(z) = -5$ e $\text{Im}(z) = 0$

e) $z = \frac{1 - \sqrt{8}i}{2} = \frac{1}{2} - \frac{2\sqrt{2}i}{2} = \frac{1}{2} - \sqrt{2}i$
$\text{Re}(z) = \frac{1}{2}$ e $\text{Im}(z) = -\sqrt{2}$

4.

a) $z = k^2 + (p-1)i$ é um número real se $\text{Im}(z) = 0$, ou seja, $p - 1 = 0 \Leftrightarrow p = 1$ e k é um qualquer número real.

b) $z = k^2 + (p-1)i$ é um imaginário se $\text{Re}(z) = 0 \wedge \text{Im}(z) \neq 0$, ou seja, $k^2 = 0 \wedge p - 1 \neq 0$
$\Leftrightarrow k = 0 \wedge p \neq 1$

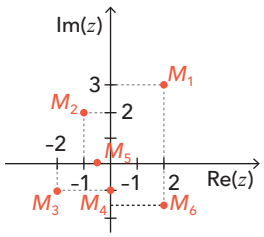
5.

a) $z^2 + 8 = 0 \Leftrightarrow z^2 = -8 \Leftrightarrow z^2 = 8i^2$
$\Leftrightarrow z = \sqrt{8}i \vee z = -\sqrt{8}i$
$\Leftrightarrow z = 2\sqrt{2}i \vee z = -2\sqrt{2}i$
C.S. $= \{2\sqrt{2}i, -2\sqrt{2}i\}$

b) $z^2 - 2z = -4 \Leftrightarrow z^2 - 2z + 4 = 0$
$\Leftrightarrow z = \frac{2 \pm \sqrt{4 - 16}}{2} \Leftrightarrow z = \frac{2 \pm i\sqrt{12}}{2}$
$\Leftrightarrow z = \frac{2 \pm 2\sqrt{3}i}{2}$
$\Leftrightarrow z = 1 + \sqrt{3}i \vee z = 1 - \sqrt{3}i$
C.S. $= \{1 + \sqrt{3}i, 1 - \sqrt{3}i\}$

c) $4z^3 - 4z^2 + 17z = 0 \Leftrightarrow z(4z^2 - 4z + 17) = 0$
$\Leftrightarrow z = 0 \vee 4z^2 - 4z + 17 = 0$
$\Leftrightarrow z = 0 \vee z = \frac{4 \pm \sqrt{16 - 272}}{8}$
$\Leftrightarrow z = 0 \vee z = \frac{2 \pm 16i}{8}$
$\Leftrightarrow z = 0 \vee z = \frac{1}{2} + 2i \vee z = \frac{1}{2} - 2i$
C.S. $= \left\{0, \frac{1}{2} + 2i, \frac{1}{2} - 2i\right\}$

6. Sejam M_1, M_2, M_3, M_4, M_5 e M_6 os afixos de z_1, z_2, z_3, z_4, z_5 e z_6, respetivamente.



7.

a) $z = 3 + 4i$
$|z| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

b) $z = 1 - i$
$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{1 + 1} = \sqrt{2}$

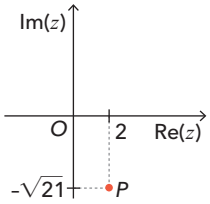
c) $z = 13i$
$|z| = \sqrt{0^2 + 13^2} = \sqrt{13^2} = 13$

d) $z = -9i$
$|z| = \sqrt{0^2 + (-9)^2} = \sqrt{9^2} = 9$

e) $z = -12i$
$|z| = \sqrt{0^2 + (-12)^2} = \sqrt{12^2} = 12$

8. Seja $z = a + bi$.
$\begin{cases} |z| = 5 \\ \text{Re}(z) = 2 \\ \text{Im}(z) < 0 \end{cases} \Leftrightarrow \begin{cases} \sqrt{a^2 + b^2} = 5 \\ a = 2 \\ b < 0 \end{cases} \Leftrightarrow \begin{cases} 2^2 + b^2 = 25 \\ a = 2 \\ b < 0 \end{cases}$
$\Leftrightarrow \begin{cases} b = \sqrt{21} \\ a = 2 \\ b < 0 \end{cases}$

Assim, $z = 2 - \sqrt{21}i$.



9.

a) $\overline{z} = -2 - 3i$

b) $\overline{z} = 1 + i$

c) $\overline{z} = -2027i$

d) $z = 5i^2 = -5$, logo $\overline{z} = -5$.

e) $z = \frac{1}{2} - \sqrt{2}i$, logo $\overline{z} = \frac{1}{2} + \sqrt{2}i$.

f) $\overline{z} = -10$

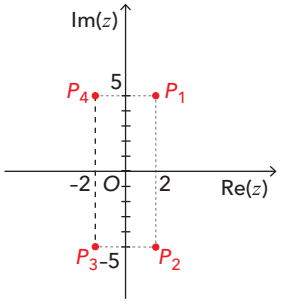
10.

a) $z = 2 + 5i$
$-z = -2 - 5i$
$\overline{z} = 2 - 5i$
$-\overline{z} = -2 + 5i$

b) Sejam P_1, P_2, P_3 e P_4 os afixos de z, $\overline{z}$, $-z$ e $-\overline{z}$, respetivamente.

$z = 2 + 5i \rightsquigarrow P_1(2, 5)$ $\quad \overline{z} = 2 - 5i \rightsquigarrow P_2(2, -5)$

$-z = -2 - 5i \rightsquigarrow P_3(-2, -5)$ $\quad -\overline{z} = -2 + 5i \rightsquigarrow P_4(-2, 5)$



c) $[P_1P_2P_3P_4]$ é um retângulo de largura 4 e comprimento 10, logo de área 40 u.a.

11.

a) $z_1 + z_2 = (4 + i) + 5i = 4 + 6i$

b) $z_1 + z_3 = (4 + i) + \left(2 - \frac{1}{2}i\right) = 6 + \frac{1}{2}i$

c) $z_2 + \overline{z}_3 = 5i + \left(2 + \frac{1}{2}i\right) = 2 + \frac{11}{2}i$

12. Seja $z = a + bi$.

a) Para $z + \overline{z} = 2$:

$(a + bi) + (a - bi) = 2 \Leftrightarrow 2a = 2$

$\Leftrightarrow a = 1$

Por exemplo, $z = 1 + 7i$.

b) Para $z + \overline{z} = 0$:

$(a + bi) + (a - bi) = 0 \Leftrightarrow 2a = 0$

$\Leftrightarrow a = 0$

Por exemplo, $z = 5i$.

c) Para $z - \overline{z} = 0$:

$(a + bi) - (a - bi) = 0 \Leftrightarrow 2bi = 0$

$\Leftrightarrow 2b = 0$

$\Leftrightarrow b = 0$

Por exemplo, $z = 3$.

13.

a) $3i \times (2 - 4i) = 6i - 12i^2 = 12 + 6i$

b) $\left(\frac{1}{2} - i\right) \times (2 + 3i) = 1 + \frac{3}{2}i - 2i - 3i^2 =$

$= 1 + 3 - \frac{1}{2}i =$

$= 4 - \frac{1}{2}i$

c) $(\sqrt{2} + i)^2 = (\sqrt{2} + i) \times (\sqrt{2} + i) =$

$= 2 + \sqrt{2}i + \sqrt{2}i + i^2 =$

$= 1 + 2\sqrt{2}i$

d) $i \times \overline{6 - 7i} = i \times (6 + 7i) = 6i + 7i^2 = -7 + 6i$

14.

a) $z = 1 + i$

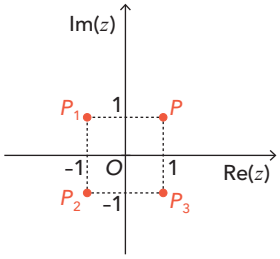
$iz = i(1 + i) = i + i^2 = -1 + i$

$i^2z = i^2(1 + i) = -1 \times (1 + i) = -1 - i$

$i^3z = -i(1 + i) = -i - i^2 = 1 - i$

Sejam P, P_1, P_2 e P_3 os afixos de z, iz, i^2z e i^3z, respetivamente.

$P(1, 1)$; $P_1(-1, 1)$; $P_2(-1, -1)$ e $P_3(1, -1)$



b) O polígono $[PP_1P_2P_3]$ é um quadrado de lado 2, logo a sua área é 4 u.a.

15.

a) $\frac{3i}{2 - 4i} = \frac{3i(2 + 4i)}{(2 - 4i)(2 + 4i)} = \frac{6i + 12i^2}{2^2 - (4i)^2} =$

$= \frac{-12 + 6i}{4 + 16} = \frac{-12 + 6i}{20} =$

$= -\frac{12}{20} + \frac{6}{20}i = -\frac{3}{5} + \frac{3}{10}i$

b) $\frac{1 - i}{1 + i} = \frac{(1 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{1 - i - i + i^2}{1^2 - i^2} =$

$= \frac{1 - 2i - 1}{1 + 1} = -\frac{2i}{2} = -i$

c) $\frac{\sqrt{3} + i}{2i} = \frac{(\sqrt{3} + i) \times i}{2i \times i} = \frac{\sqrt{3}i + i^2}{2i^2} = \frac{-1 + \sqrt{3}i}{-2} =$

$= \frac{1}{2} - \frac{\sqrt{3}}{2}i$

d) $\frac{1}{5 - i} = \frac{5 + i}{(5 - i)(5 + i)} = \frac{5 + i}{5^2 - i^2} = \frac{5 + i}{26} = \frac{5}{26} + \frac{1}{26}i$

16.

a) $i^{101} = i^1 = i$

101	4
1	25

b) $i^{2014} = i^2 = -1$

2014	4
2	503

c) $i^{23} \times (i^5 + 5i^{11}) = i^3 \times (i^2 + 5i^3) =$

$= -i(-1 - 5i) =$

$= i + 5i^2 = -5 + i$

d) $2i^{4n+1} - 3i^{4n+3} + (2i)^8 = 2 \times i^1 - 3i^3 + 256i^8 =$

$= 2i + 3i + 256 =$

$= 256 + 5i$

17. $i^{9n} + i^{-9n} = i^{8n+n} + i^{-8n-n} =$

$= i^n + i^{-n} =$

$= i^n + \frac{1}{i^n} =$

$= \frac{i^{2n} + 1}{i^n} =$

$= \frac{(i^2)^n + 1}{i^n} =$

$= \frac{(-1)^n + 1}{i^n}$

Como n é ímpar, então, $(-1)^n = -1$, logo

$\frac{(-1)^n + 1}{i^n} = \frac{-1 + 1}{i^n} = 0.$

18.

a)

```
print("Potência de base i e expoente n")
n=int(input("n?"))
r=n%4
if r==0:
  print("1")
elif r==1:
  print ("i")
elif r==2:
  print ("-1")
else:
  print ("-i")
```

```
Potência de base i e expoente n
n?245
i
```

b)

```
print("Potência de base i e expoente n")
n=int(input("n?"))
r=n%4
if r==0:
  print("1")
elif r==1:
  print ("i")
elif r==2:
  print ("-1")
else:
  print ("-i")
```

```
Potência de base i e expoente n
n?338
-1
```

c)

```
print("Potência de base i e expoente n")
n=int(input("n?"))
r=n%4
if r==0:
  print("1")
elif r==1:
  print ("i")
elif r==2:
  print ("-1")
else:
  print ("-i")
```

```
Potência de base i e expoente n
n?439
-i
```

d)

```
print("Potência de base i e expoente n")
n=int(input("n?"))
r=n%4
if r==0:
  print("1")
elif r==1:
  print ("i")
elif r==2:
  print ("-1")
else:
  print ("-i")
```

```
Potência de base i e expoente n
n?560
1
```

19.

a) $(2-6i)^2 = 4 - 24i + 36i^2 = 4 - 36 - 24i = -32 - 24i$

b) $(1+i)^5 = (1+i)^2 \times (1+i)^2 \times (1+i) =$

$= (1 + 2i + i^2)(1 + 2i + i^2)(1 + i) =$

$= (2i)(2i)(1 + i) =$

$= 4i^2 (1 + i) =$

$= -4(1 + i) =$

$= -4 - 4i$

c) $(1+i)^{-5} = \frac{1}{(1+i)^5} = \frac{1}{-4-4i} =$

(pelos cálculos efetuados na alínea anterior)

$= \frac{-4+4i}{(-4-4i)(-4+4i)} = \frac{-4+4i}{(-4)^2-(4i)^2} =$

$= \frac{-4+4i}{16+16} = -\frac{4}{32} + \frac{4}{32}i =$

$= -\frac{1}{8} + \frac{1}{8}i$

d) $(3-i)^{-4} = \frac{1}{(3-i)^4} = \frac{1}{((3-i)^2)^2} =$

$= \frac{1}{(9-6i+i^2)^2} = \frac{1}{(8-6i)^2} =$

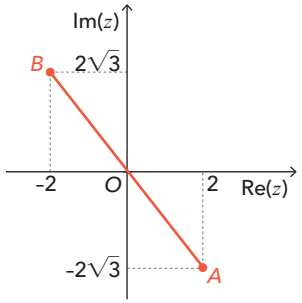
$= \frac{1}{64-96i+36i^2} = \frac{1}{28-96i} =$

$= \frac{28+96i}{(28-96i)(28+96i)} = \frac{28+96i}{28^2-(96i)^2} =$

$= \frac{28+96i}{784+9216} = \frac{28}{10\,000} + \frac{96}{10\,000}i =$

$= \frac{7}{2500} + \frac{6}{625}i$

20. Como $i^{50} = i^2 = -1$, tem-se que $z_3 = i^{50} \times z_1 = -z_1$, isto é, z_3 é o simétrico de z_1 e, portanto, o seu afixo, B, será simétrico do afixo de z_1, A, em relação à origem do referencial. Portanto:



$\overline{AB} = |z_1| + |z_3| = 4 + 4 = 8$ u.c.

Cálculo auxiliar

$|z_1| = |z_3| = \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{4 + 12} = 4$

21.

a) $w = \overline{-\frac{1}{2}i^2 + 3i} - z =$

$= \overline{\frac{1}{2} + 3i} - (-1 - \sqrt{2}i) =$

$= \frac{1}{2} - 3i + 1 + \sqrt{2}i =$

$= \frac{3}{2} + (\sqrt{2} - 3)i$

Logo, $\text{Re}(w) = \frac{3}{2}$ e $\text{Im}(w) = \sqrt{2} - 3$.

b) $\overline{2z+3w} = \overline{2(-1-\sqrt{2}i)+3\left(\frac{3}{2}+(\sqrt{2}-3)i\right)} =$

$= \overline{-2-2\sqrt{2}i+\frac{9}{2}+3\sqrt{2}i-9i} =$

$= \overline{\frac{5}{2}+(-9+\sqrt{2})i} =$

$= \frac{5}{2}-(-9+\sqrt{2})i =$

$= \frac{5}{2}+(9-\sqrt{2})i$

22.

a) $\frac{3+5i}{1+7i} = \frac{(3+5i)(1-7i)}{(1+7i)(1-7i)} =$

$= \frac{3-21i+5i-35i^2}{1^2-(7i)^2} =$

$= \frac{3-16i+35}{1-49i^2} =$

$= \frac{38-16i}{50} =$

$= \frac{38}{50}-\frac{16}{50}i =$

$= \frac{19}{25}-\frac{8}{25}i$

Assim, $\text{Re}\left(\frac{3+5i}{1+7i}\right) = \frac{19}{25}$ e $\text{Im}\left(\frac{3+5i}{1+7i}\right) = -\frac{8}{25}$.

b) $(1-\sqrt{2}i)^3 = (1-\sqrt{2}i)^2 \times (1-\sqrt{2}i) =$

$= (1-2\sqrt{2}i+(\sqrt{2}i)^2)(1-\sqrt{2}i) =$

$= (1-2\sqrt{2}i+2i^2)(1-\sqrt{2}i) =$

$= (-1-2\sqrt{2}i)(1-\sqrt{2}i) =$

$= -1+\sqrt{2}i-2\sqrt{2}i+2\times(\sqrt{2})^2i^2 =$

$= -1-\sqrt{2}i-4 =$

$= -5-\sqrt{2}i$

Assim, $\text{Re}((1-\sqrt{2}i)^3) = -5$ e $\text{Im}((1-\sqrt{2}i)^3) = -\sqrt{2}$.

23.

a) $z = \frac{k+i}{k-i} = \frac{(k+i)\times(k+i)}{(k-i)(k+i)} = \frac{k^2+2ik+i^2}{k^2-i^2} =$

$= \frac{(k^2-1)+2ki}{k^2+1} =$

$= \frac{k^2-1}{k^2+1}+\frac{2k}{k^2+1}i$

Para que z seja um número real terá de se verificar $\text{Im}(z) = 0$.

Assim:

$\frac{2k}{k^2+1} = 0 \Leftrightarrow 2k = 0 \wedge \underbrace{k^2+1 \neq 0}_{\text{condição universal em } \mathbb{R}}$

$\Leftrightarrow k = 0$

b) Para que z seja um imaginário puro, terá de se verificar $\text{Re}(z) = 0 \wedge \text{Im}(z) \neq 0$.

Assim:

$\frac{k^2-1}{k^2+1} = 0 \wedge \frac{2k}{k^2+1} \neq 0$

$\Leftrightarrow k^2-1 = 0 \wedge \underbrace{k^2+1 \neq 0}_{\text{condição universal em } \mathbb{R}} \wedge 2k \neq 0$

$\Leftrightarrow k^2 = 1 \wedge k \neq 0$

$\Leftrightarrow k = 1 \vee k = -1$

24.

a) $\frac{z_1}{z_2} = \frac{1+i}{4i} = \frac{(1+i)\times i}{4i\times i} = \frac{i+i^2}{4i^2} =$

$= \frac{-1+i}{-4} = \frac{1}{4}-\frac{1}{4}i$

b) $(z_1)^3 \times z_3 = (1+i)^3 \times (2-3i) =$

$= (1+i)^2(1+i)(2-3i) =$

$= (1+2i+i^2)(1+i)(2-3i) =$

$= 2i\times(1+i)(2-3i) =$

$= (2i+2i^2)(2-3i) =$

$= (-2+2i)(2-3i) =$

$= -4+6i+4i-6i^2 =$

$= -4+10i+6 =$

$= 2+10i$

c) $\frac{\overline{z_2}-z_3}{i} = \frac{(-4i)-(2-3i)}{i} = \frac{-4i-2+3i}{i} =$

$= \frac{-2-i}{i} = \frac{(-2-i)\times i}{i^2} =$

$= \frac{-2i-i^2}{-1} = \frac{1-2i}{-1} =$

$= -1+2i$

d) $z_2+\frac{\overline{z_3}}{5i^{8n+3}} = 4i+\frac{2+3i}{5\times i^3} =$

$= 4i+\frac{2+3i}{-5i} = 4i+\frac{(2+3i)\times i}{-5i\times i} =$

$= 4i+\frac{2i+3i^2}{-5i^2} = 4i+\frac{-3+2i}{5} =$

$= 4i-\frac{3}{5}+\frac{2}{5}i = -\frac{3}{5}+\frac{22}{5}i$

e) $z_1-\frac{i}{z_3} = 1+i-\frac{i}{2-3i} = 1+i-\frac{i(2+3i)}{(2-3i)(2+3i)} =$

$= 1+i-\frac{2i+3i^2}{2^2-(3i)^2} = 1+i-\frac{-3+2i}{4+9} =$

$= 1+i+\frac{3}{13}-\frac{2}{13}i = \frac{16}{13}+\frac{11}{13}i$

25.

a) $-i+i^2z = i^4z+1 \Leftrightarrow -i-z = z+1$

$\Leftrightarrow 2z = -1-i$

$\Leftrightarrow z = -\frac{1}{2}-\frac{1}{2}i$

C.S. $= \left\{-\frac{1}{2}-\frac{1}{2}i\right\}$

b) $(3-i)+(1+i)z = 5i \Leftrightarrow (1+i)z = 5i-3+i$

$\Leftrightarrow z = \frac{-3+6i}{1+i}$

$\Leftrightarrow z = \frac{(-3+6i)(1-i)}{(1+i)(1-i)}$

$\Leftrightarrow z = \frac{-3+3i+6i-6i^2}{1^2-i^2}$

$\Leftrightarrow z = \frac{3+9i}{1+1}$

$\Leftrightarrow z = \frac{3}{2}+\frac{9}{2}i$

C.S. $= \left\{\frac{3}{2}+\frac{9}{2}i\right\}$

c) $z - iz = \frac{2-i}{3i} \Leftrightarrow z(1-i) = \frac{2-i}{3i}$

$\Leftrightarrow z = \frac{2-i}{3i(1-i)}$

$\Leftrightarrow z = \frac{2-i}{3i-3i^2}$

$\Leftrightarrow z = \frac{2-i}{3+3i}$

$\Leftrightarrow z = \frac{(2-i)(3-3i)}{(3+3i)(3-3i)}$

$\Leftrightarrow z = \frac{6-6i-3i+3i^2}{3^2-(3i)^2}$

$\Leftrightarrow z = \frac{6-3-9i}{18}$

$\Leftrightarrow z = \frac{1}{6} - \frac{1}{2}i$

C.S. $= \left\{\frac{1}{6} - \frac{1}{2}i\right\}$

26.

a) **i.** $x^3 - 3x^2 + 3x - 1 = 0$

$\Leftrightarrow (x-1)(x^2-2x+1) = 0$

Cálculo auxiliar

	1	−3	3	−1
1		1	−2	1
	1	−2	1	0

$\Leftrightarrow x - 1 = 0 \vee x^2 - 2x + 1 = 0$

$\Leftrightarrow x = 1 \vee x = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 1}}{2 \times 1}$

$\Leftrightarrow x = 1 \vee x = \frac{2 \pm 0}{2}$

$\Leftrightarrow x = 1 \vee x = 1 \vee x = 1$

C.S. $= \{1\}$

ii. $x^3 - 5x^2 + 8x - 4 = 0$

$\Leftrightarrow (x-1)(x^2-4x+4) = 0$

Cálculo auxiliar

	1	−5	8	−4
1		1	−4	4
	1	−4	4	0

$\Leftrightarrow x - 1 = 0 \vee x^2 - 4x + 4 = 0$

$\Leftrightarrow x = 1 \vee x = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 4}}{2 \times 1}$

$\Leftrightarrow x = 1 \vee x = \frac{4 \pm 0}{2}$

$\Leftrightarrow x = 1 \vee x = 2 \vee x = 2$

C.S. $= \{1, 2\}$

iii. $x^3 - 6x^2 + 11x - 6 = 0$

$\Leftrightarrow (x-1)(x^2-5x+6) = 0$

Cálculo auxiliar

	1	−6	11	−6
1		1	−5	6
	1	−5	6	0

$\Leftrightarrow x - 1 = 0 \vee x^2 - 5x + 6 = 0$

$\Leftrightarrow x = 1 \vee x = \frac{5 \pm \sqrt{25 - 4 \times 1 \times 6}}{2 \times 1}$

$\Leftrightarrow x = 1 \vee x = \frac{5 \pm 1}{2}$

$\Leftrightarrow x = 1 \vee x = 3 \vee x = 2$

C.S. $= \{1, 2, 3\}$

iv. $x^3 - x^2 + 4x - 4 = 0$

$\Leftrightarrow (x-1)(x^2+4) = 0$

Cálculo auxiliar

	1	−1	4	−4
1		1	0	4
	1	0	4	0

$\Leftrightarrow x - 1 = 0 \vee x^2 + 4 = 0$

$\Leftrightarrow x = 1 \vee x^2 = -4$

$\Leftrightarrow x = 1 \vee x = \pm\sqrt{-4}$

$\Leftrightarrow x = 1 \vee x = 2i \vee x = -2i$

C.S. $= \{1, 2i, -2i\}$

v. $x^3 - 5x^2 + 17x - 13 = 0$

$\Leftrightarrow (x-1)(x^2-4x+13) = 0$

Cálculo auxiliar

	1	−5	17	−13
1		1	−4	13
	1	−4	13	0

$\Leftrightarrow x - 1 = 0 \vee x^2 - 4x + 13 = 0$

$\Leftrightarrow x = 1 \vee x = \frac{4 \pm \sqrt{16 - 4 \times 1 \times 13}}{2 \times 1}$

$\Leftrightarrow x = 1 \vee x = \frac{4 \pm \sqrt{-36}}{2}$

$\Leftrightarrow x = 1 \vee x = \frac{4 \pm 6i}{2}$

$\Leftrightarrow x = 1 \vee x = 2 + 3i \vee x = 2 - 3i$

C.S. $= \{1, 2 + 3i, 2 - 3i\}$

b) **i.** Uma solução real tripla.

ii. Duas soluções reais (uma simples e uma dupla).

iii. Três soluções reais.

iv. Uma solução real e duas soluções não reais, complexas conjugadas.

v. Uma solução real e duas soluções não reais, complexas conjugadas.

c) As equações do 3º grau, de coeficientes reais, têm uma, duas ou três soluções em $\mathbb{C}$, sendo que, pelo menos, uma delas é sempre real e as outras duas, se não forem reais, são complexas conjugadas.

27.

a)

```
print('ax^2+bx+c=0')
a=float(input('a?'))
b=float(input('b?'))
c=float(input('c?'))
delta=b**2-4*a*c
if delta==0:
  x1=(-b-(delta)**(1/2))/(2*a)
  print('Solução: ',x1)
else:
  x1=(-b-(delta)**(1/2))/(2*a)
  x2=(-b+(delta)**(1/2))/(2*a)
  print('Duas soluções:',x1,' e ',x2)
```

```
ax^2+bx+c=0
a?1
b?2
c?1
Solução:  -1.0
```

b)

```
print('ax^2+bx+c=0')
a=float(input('a?'))
b=float(input('b?'))
c=float(input('c?'))
delta=b**2-4*a*c
if delta==0:
  x1=(-b-(delta)**(1/2))/(2*a)
  print('Solução: ',x1)
else:
  x1=(-b-(delta)**(1/2))/(2*a)
  x2=(-b+(delta)**(1/2))/(2*a)
  print('Duas soluções:',x1,' e ',x2)
```

```
ax^2+bx+c=0
a?1
b?-7
c?12
Duas soluções: 3.0  e  4.0
```

c)

```
print('ax^2+bx+c=0')
a=float(input('a?'))
b=float(input('b?'))
c=float(input('c?'))
delta=b**2-4*a*c
if delta==0:
  x1=(-b-(delta)**(1/2))/(2*a)
  print('Solução: ',x1)
else:
  x1=(-b-(delta)**(1/2))/(2*a)
  x2=(-b+(delta)**(1/2))/(2*a)
  print('Duas soluções:',x1,' e ',x2)
```

```
ax^2+bx+c=0
a?1
b?-6
c?13
Duas soluções: (3-2j)  e  (3+2j)
```

Aprende fazendo (1) – páginas 28 e 29

28.

a) $z + w = (1 - 2i) + (4 + 5i) = (1 + 4) + (-2 + 5)i = 5 + 3i$

b) $3z - 2w = 3(1 - 2i) - 2(4 + 5i) =$
$= 3 - 6i - 8 - 10i =$
$= -5 - 16i$

c) $z \times w = (1 - 2i) \times (4 + 5i) = 4 + 5i - 8i - 10i^2 =$
$= 4 + 10 - 3i =$
$= 14 - 3i$

d) $\frac{z}{w} = \frac{1 - 2i}{4 + 5i} = \frac{(1 - 2i)(4 - 5i)}{(4 + 5i)(4 - 5i)} = \frac{4 - 5i - 8i + 10i^2}{4^2 - (5i)^2} =$
$= \frac{4 - 13i - 10}{16 + 25} = \frac{-6 - 13i}{41} = -\frac{6}{41} - \frac{13}{41}i$

e) $\frac{1}{z} = \frac{1}{1 - 2i} = \frac{1 + 2i}{(1 - 2i)(1 + 2i)} = \frac{1 + 2i}{1^2 - (2i)^2} = \frac{1 + 2i}{1 + 4} =$
$= \frac{1}{5} + \frac{2}{5}i$

f) $z^2 - \frac{i}{\bar{z}} = (1 - 2i)^2 - \frac{i}{1 + 2i} =$
$= 1 - 4i + 4i^2 - \frac{i(1 - 2i)}{(1 + 2i)(1 - 2i)} =$
$= 1 - 4i - 4 - \frac{i - 2i^2}{1^2 - (2i)^2} =$
$= -3 - 4i - \frac{2 + i}{1 + 4} =$
$= -3 - 4i - \frac{2}{5} - \frac{1}{5}i =$
$= -\frac{17}{5} - \frac{21}{5}i$

29.

a) $(1 - i)z - 2i = i \Leftrightarrow (1 - i)z = 3i \Leftrightarrow z = \frac{3i}{1 - i}$
$\Leftrightarrow z = \frac{3i(1 + i)}{(1 - i)(1 + i)}$
$\Leftrightarrow z = \frac{3i + 3i^2}{1 - i^2}$
$\Leftrightarrow z = \frac{-3 + 3i}{2}$
$\Leftrightarrow z = -\frac{3}{2} + \frac{3}{2}i$

C.S. $= \left\{-\frac{3}{2} + \frac{3}{2}i\right\}$

b) $2z - iz = \frac{1 - i}{3i} \Leftrightarrow z(2 - i) = \frac{1 - i}{3i}$
$\Leftrightarrow z = \frac{1 - i}{3i(2 - i)}$
$\Leftrightarrow z = \frac{1 - i}{6i - 3i^2}$
$\Leftrightarrow z = \frac{1 - i}{3 + 6i}$
$\Leftrightarrow z = \frac{(1 - i)(3 - 6i)}{(3 + 6i)(3 - 6i)}$
$\Leftrightarrow z = \frac{3 - 6i - 3i + 6i^2}{3^2 - (6i)^2}$
$\Leftrightarrow z = \frac{3 - 6 - 9i}{9 - 36i^2}$
$\Leftrightarrow z = \frac{-3 - 9i}{9 + 36}$
$\Leftrightarrow z = -\frac{3}{45} - \frac{9}{45}i$
$\Leftrightarrow z = -\frac{1}{15} - \frac{1}{5}i$

C.S. $= \left\{-\frac{1}{15} - \frac{1}{5}i\right\}$

c) $i^3z + 1 = i^2z - i \Leftrightarrow -iz + 1 = -z - i$

$\Leftrightarrow -iz + z = -1 - i$

$\Leftrightarrow z(-i + 1) = -1 - i$

$\Leftrightarrow z = \frac{-1 - i}{1 - i}$

$\Leftrightarrow z = \frac{(-1 - i)(1 + i)}{(1 - i)(1 + i)}$

$\Leftrightarrow z = \frac{-1 - i - i - i^2}{1^2 - i^2}$

$\Leftrightarrow z = \frac{-1 - 2i + 1}{2}$

$\Leftrightarrow z = -i$

C.S. $= \{-i\}$

d) $z^2 - 2z + 10 = 0 \Leftrightarrow z = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 10}}{2 \times 1}$

$\Leftrightarrow z = \frac{2 \pm \sqrt{-36}}{2 \times 1}$

$\Leftrightarrow z = \frac{2 \pm 6i}{2}$

$\Leftrightarrow z = 1 + 3i \vee z = 1 - 3i$

C.S. $= \{1 + 3i, 1 - 3i\}$

e) $4z^3 - 4z^2 + 5z = 0 \Leftrightarrow z(4z^2 - 4z + 5) = 0$

$\Leftrightarrow z = 0 \vee 4z^2 - 4z + 5 = 0$

$\Leftrightarrow z = 0 \vee z = \frac{4 \pm \sqrt{16 - 4 \times 4 \times 5}}{2 \times 4}$

$\Leftrightarrow z = 0 \vee z = \frac{4 \pm \sqrt{-64}}{8}$

$\Leftrightarrow z = 0 \vee z = \frac{4 \pm 8i}{8}$

$\Leftrightarrow z = 0 \vee z = \frac{1}{2} + i \vee z = \frac{1}{2} - i$

C.S. $= \left\{0, \frac{1}{2} + i, \frac{1}{2} - i\right\}$

30. Opção (B)

$z = \sqrt{3}i^{18} + i^{27} = \sqrt{3}i^2 + i^3 = -\sqrt{3} - i$

$|z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3 + 1} = 2$

31.

a) $z = \frac{2 + 12i}{1 - 3i} = \frac{(2 + 12i)(1 + 3i)}{(1 - 3i)(1 + 3i)} = \frac{2 + 6i + 12i + 36i^2}{1^2 - (3i)^2} =$

$= \frac{2 + 18i - 36}{1 + 9} =$

$= \frac{-34}{10} + \frac{18}{10}i =$

$= -\frac{17}{5} + \frac{9}{5}i$

$\text{Re}(z) = -\frac{17}{5}$ e $\text{Im}(z) = \frac{9}{5}$

b) $w = \frac{i}{5 - 10i} = \frac{i(5 + 10i)}{(5 - 10i)(5 + 10i)} =$

$= \frac{5i + 10i^2}{5^2 - (10i)^2} =$

$= \frac{-10 + 5i}{25 + 100} =$

$= -\frac{10}{125} + \frac{5}{125}i =$

$= -\frac{2}{25} + \frac{1}{25}i$

$\text{Re}(w) = -\frac{2}{25}$ e $\text{Im}(w) = \frac{1}{25}$

c) Seja $z_1 = i^3z$.

$i^3z = i^3 \times \frac{2 + 12i}{1 - 3i} =$

$= -i \times \left(-\frac{17}{5} + \frac{9}{5}i\right)$ (pelos cálculos efetuados na alínea **a)**)

$= \frac{17}{5}i - \frac{9}{5}i^2 =$

$= \frac{9}{5} + \frac{17}{5}i$

$\text{Re}(i^3z) = \frac{9}{5}$ e $\text{Im}(i^3z) = \frac{17}{5}$

32. Opção (B)

Seja $z = a + bi$, com $b \neq 0$. Então:

$\overline{z - 1} = \overline{(a + bi) - 1} = \overline{(a - 1) + bi} =$

$= (a - 1) - bi =$

$= (a - bi) - 1 =$

$= \overline{z} - 1$

33. Opção (B)

$z^7 = (bi)^7 = b^7\, i^7 = b^7 \times (-i) = -b^2\, i$

Como $b < 0$, então $-b^7 > 0$. Logo, o afixo de z^7 encontra-se na parte positiva do eixo imaginário.

34. Opção (A)

Seja $w = x + yi$ um número complexo não nulo.

$\frac{w \times \overline{w}}{2} = \frac{(x + yi) \times (x - yi)}{2} = \frac{x^2 + y^2}{8} \in \mathbb{R}^+$

Logo, o afixo de $\frac{w \times \overline{w}}{2}$ encontra-se na parte positiva do eixo real.

35. $z^3 - z^2 + 9z - 9 = 0 \Leftrightarrow (z - 1) \times (z^2 + 9) = 0$

$\Leftrightarrow z - 1 = 0 \vee z^2 + 9 = 0$

$\Leftrightarrow z = 1 \vee z^2 = -9$

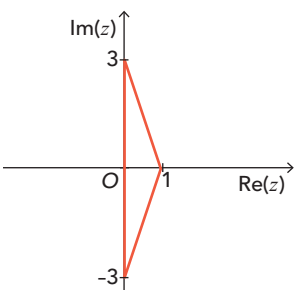
$\Leftrightarrow z = 1 \vee z = 3i \vee z = -3i$

Cálculo auxiliar

	1	−1	9	−9
1		1	0	9
	1	0	9	0

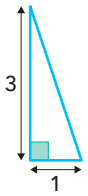
$z^3 + z^2 + 9z - 9 = (z - 1)(z^2 + 9)$

Representando os afixos dos números complexos 1, $3i$ e $-3i$ obtemos o triângulo representado na figura.



O perímetro deste triângulo é $\left(6 + 2\sqrt{10}\right)$ u.c.

Cálculo auxiliar



$\ell^2 = 1^2 + 3^2$, logo, $\ell = \sqrt{10}$, $\ell > 0$

36. Opção (B)

$i^{2n+3} = i^{2n} \times i^3 = (i^2)^n \times (-i) = (-1)^n \times (-i)$

Se n é par:

$(-1)^n \times (-i) = 1 \times (-i) = -i$

Se n é ímpar:

$(-1)^n \times (-i) = -1 \times (-i) = i$

37. Opção (C)

$$\frac{z}{w} = \frac{a+bi}{1+i} = \frac{(a+bi)(1-i)}{(1+i)(1-i)} = \frac{a-ai+bi-bi^2}{1^2-i^2} =$$

$$= \frac{a+b+(-a+b)i}{2} = \frac{a+b}{2} + \frac{b-a}{2}i$$

Assim, $\text{Re}\left(\frac{z}{w}\right) = \frac{a+b}{2}$ e $\text{Im}\left(\frac{z}{w}\right) = \frac{b-a}{2}$.

$z \times w = (a+bi)(1+i) = a + ai + bi + bi^2 =$

$= (a-b) + (a+b)i$

Assim, $\text{Im}(z \times w) = a + b$.

$z^2 = (a+bi)^2 = a^2 + 2abi + (bi)^2 = a^2 - b^2 + 2abi$

Assim, $\text{Re}(z^2) = a^2 - b^2$.

38.

a) Para $\alpha = \frac{5\pi}{4}$, tem-se:

$$z = \text{tg}\left(\frac{5\pi}{4}\right) + 2\cos\left(\frac{5\pi}{4}\right)i =$$

$$= 1 + 2 \times \left(-\frac{\sqrt{2}}{2}i\right) =$$

$$= 1 - \sqrt{2}i$$

Assim, para $z = 1 - \sqrt{2}i$ ser solução de equação $\frac{z \times \bar{z} + i^{2027}}{1-2i} = 1 + i$ tem que se verificar

$$\frac{(1-\sqrt{2}i) \times \overline{(1-\sqrt{2}i)} + i^{2027}}{1-2i} = 1 + i.$$

Cálculo auxiliar

2027	4
3	506

$$\frac{(1-\sqrt{2}i) \times \overline{(1-\sqrt{2}i)} + i^{2027}}{1-2i} =$$

$$= \frac{(1-\sqrt{2}i) \times (1+\sqrt{2}i) + i^3}{1-2i} =$$

$$= \frac{1^2 - (\sqrt{2}i)^2 - i}{1-2i} =$$

$$= \frac{1+2-i}{1-2i} =$$

$$= \frac{(3-i) \times (1+2i)}{(1-2i)(1+2i)} =$$

$$= \frac{3+6i-i-2i^2}{1^2-(2i)^2} =$$

$$= \frac{5+5i}{1+4} =$$

$$= 1 + i$$

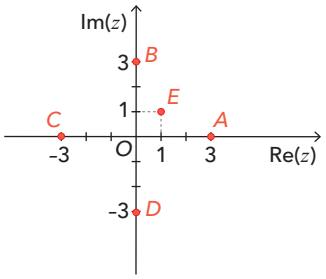
b) Para $z = \text{tg}\,\alpha + 2\cos\alpha i$ ser um número real tem que $\text{Im}(z) = 0$, isto é, $2\cos\alpha = 0$.

No entanto, se $\cos\alpha = 0$, então $\text{tg}\,\alpha$, que é $\text{Re}(z)$, não se encontra definida e, portanto, o número complexo z não existe nestas condições.

Capítulo 2 – Números complexos na forma trigonométrica

Tarefa – O radar – página 30

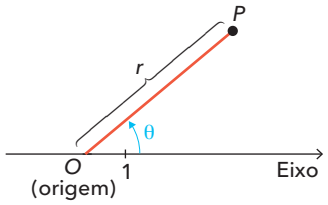
a)



b) **i.** e **ii.**

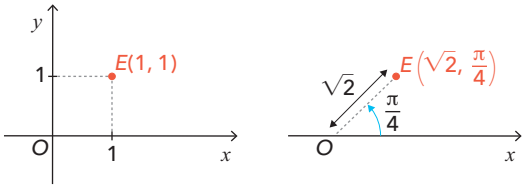
Ponto	Distância r	Ângulo α (em radianos)
A	3	0
B	3	$\frac{\pi}{2}$
C	3	π
D	3	$\frac{3\pi}{2}$
E	$\sqrt{2}$	$\frac{\pi}{4}$

c) Sim. Apesar de trabalharmos exclusivamente com referenciais cartesianos e, portanto, com coordenadas cartesianas, existem outros sistemas de coordenadas, como é o caso do **sistema de coordenadas polares**. Neste sistema, considera-se um eixo orientado (eixo polar) e um ponto O nesse eixo (origem ou polo).



Neste referencial, qualquer ponto P do plano fica definido pela distância deste à origem, r, e pela medida θ, em radianos, do ângulo orientado de vértice O, que tem o lado origem sobre o eixo e o lado extremidade $\dot{O}P$. Assim, as coordenadas polares do ponto, neste referencial, são (r, θ).

Por exemplo, o ponto E da alínea a) tem coordenadas cartesianas (1, 1) e coordenadas polares $\left(\sqrt{2}, \frac{\pi}{4}\right)$.



Exercícios de margem – páginas 34 a 67

39.

a) $2i = 2e^{i\frac{\pi}{2}}$

b) $-3i = 3e^{i\left(-\frac{\pi}{2}\right)}$

c) $2026 = 2026e^{i0}$

d) $-2027 = 2027e^{i\pi}$

e) $2 + 2i = 2\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) =$

$= 2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\,\text{sen}\left(\frac{\pi}{4}\right)\right) =$

$= 2\sqrt{2}e^{i\frac{\pi}{4}}$

Cálculo auxiliar

$|2 + 2i| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

40.

a) $\rho = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = 2$

$\text{tg}\,\theta = \frac{\sqrt{3}}{1} \wedge \theta \in 1.º\,Q \Leftrightarrow \text{tg}\,\theta = \sqrt{3} \wedge \theta \in 1.º\,Q$

Por exemplo, $\theta = \frac{\pi}{3}$.

Assim, $1 + \sqrt{3}i = 2e^{i\frac{\pi}{3}}$.

b) $\rho = \sqrt{(-\sqrt{2})^2 + (\sqrt{6})^2} = \sqrt{2 + 6} = \sqrt{8} = 2\sqrt{2}$

$\text{tg}\,\theta = \frac{\sqrt{6}}{-\sqrt{2}} \wedge \theta \in 2.º\,Q \Leftrightarrow \text{tg}\,\theta = -\sqrt{3} \wedge \theta \in 2.º\,Q$

Por exemplo, $\theta = \frac{2\pi}{3}$.

Assim, $-\sqrt{2} + \sqrt{6}i = 2\sqrt{2}e^{i\frac{2\pi}{3}}$.

c) $\rho = \sqrt{(-3)^2 + (-\sqrt{3})^2} = \sqrt{9 + 3} = \sqrt{12} = 2\sqrt{3}$

$\text{tg}\,\theta = \frac{-\sqrt{3}}{-3} \wedge \theta \in 3.º\,Q \Leftrightarrow \text{tg}\,\theta = \frac{\sqrt{3}}{3} \wedge \theta \in 3.º\,Q$

Por exemplo, $\theta = \frac{7\pi}{6}$.

Assim, $-3 - \sqrt{3}i = 2\sqrt{3}e^{i\frac{7\pi}{6}}$.

d) $\rho = \sqrt{(-\sqrt{12})^2 + 2^2} = \sqrt{12 + 4} = 4$

$\text{tg}\,\theta = \frac{2}{-\sqrt{12}} \wedge \theta \in 2.º\,Q \Leftrightarrow \text{tg}\,\theta = -\frac{\sqrt{3}}{3} \wedge \theta \in 2.º\,Q$

Por exemplo, $\theta = \frac{5\pi}{6}$.

Assim, $-\sqrt{12} + 2i = 4e^{i\frac{5\pi}{6}}$.

e) $\rho = \sqrt{2^2 + (-2)^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$

$\text{tg}\,\theta = \frac{-2}{2} \wedge \theta \in 4.º\,Q \Leftrightarrow \text{tg}\,\theta = -1 \wedge \theta \in 4.º\,Q$

Por exemplo, $\theta = -\frac{\pi}{4}$.

Assim, $2 - 2i = 2\sqrt{2}e^{i\left(-\frac{\pi}{4}\right)}$.

f) $3 + i + 3i^2 = 3 + i - 3 = i = e^{i\frac{\pi}{2}}$

g) $(1 - i)^2 = 1 - 2i + i^2 = 1 - 2i - 1 = -2i = 2e^{i\frac{3\pi}{2}}$

41.

a) $e^{i\frac{\pi}{3}} = \cos\left(\frac{\pi}{3}\right) + i\,\text{sen}\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

b) $\sqrt{2}e^{i\frac{\pi}{4}} = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\,\text{sen}\left(\frac{\pi}{4}\right)\right) =$

$= \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) =$

$= 1 + i$

c) $4e^{i\left(-\frac{\pi}{6}\right)} = 4\left(\cos\left(-\frac{\pi}{6}\right) + i\,\text{sen}\left(-\frac{\pi}{6}\right)\right) =$

$= 4\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) =$

$= 2\sqrt{3} - 2i$

d) $3e^{i\frac{\pi}{2}} = 3i$

e) $8e^{i2026\pi} = 8$

f) $8e^{i2027\pi} = -8$

g) $9e^{i\frac{3\pi}{2}} = -9i$

h) $e^{i\frac{7\pi}{4}} = \cos\frac{7\pi}{4} + i\,\text{sen}\,\frac{7\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

42.

a) $|z| = 3 \wedge \arg(z) = -\frac{\pi}{6}$

$|w| = 3 \wedge \arg(w) = \frac{11\pi}{6} = -\frac{\pi}{6} + 2\pi$

z e w são iguais, pois têm módulos iguais e os argumentos diferem de 2π.

b) $|z| = \sqrt{8} = 2\sqrt{2} \wedge \arg(z) = \frac{\pi}{3}$

$|w| = 2\sqrt{2} \wedge \arg(w) = \frac{13\pi}{3} = \frac{\pi}{3} + 4\pi$

z e w são iguais, pois têm módulos iguais e os argumentos diferem de 4π.

c) $|z| = \sqrt{12} = 2\sqrt{3} \wedge \arg(z) = \frac{\pi}{7}$

$|w| = 3\sqrt{2} \wedge \arg(w) = \frac{\pi}{7}$

z e w não são iguais, pois têm módulos diferentes.

43. $z = \rho e^{i\left(\theta + \frac{\pi}{3}\right)}$

$w = 1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$

$z = w \Leftrightarrow \rho e^{i\left(\theta + \frac{\pi}{3}\right)} = \sqrt{2}e^{i\frac{\pi}{4}}$

$\Leftrightarrow \rho = \sqrt{2} \wedge \theta + \frac{\pi}{3} = \frac{\pi}{4} + 2k\pi,\ k \in \mathbb{Z}$

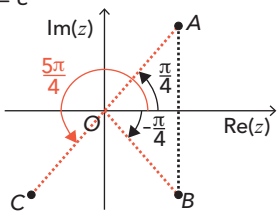
$\Leftrightarrow \rho = \sqrt{2} \wedge \theta = -\frac{\pi}{12} + 2k\pi,\ k \in \mathbb{Z}$

44. Sejam A o afixo de z, B o afixo de $\overline{z}$ e C o afixo de $-z$.

a) $z = e^{i\frac{\pi}{4}}$

$\overline{z} = e^{i\left(-\frac{\pi}{4}\right)}$

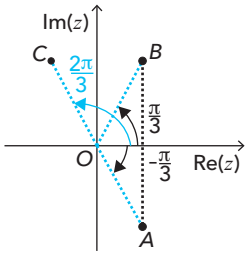
$-z = e^{i\left(\frac{\pi}{4}+\pi\right)} = e^{i\frac{5\pi}{4}}$



b) $z = 2e^{i\left(-\frac{\pi}{3}\right)}$

$\overline{z} = 2e^{i\frac{\pi}{3}}$

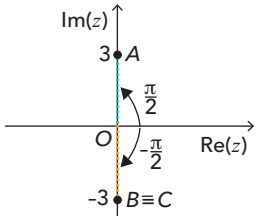
$-z = 2e^{i\left(-\frac{\pi}{3}+\pi\right)} = 2e^{i\frac{2\pi}{3}}$



c) $z = 3i = 3e^{i\frac{\pi}{2}}$

$\overline{z} = 3e^{i\left(-\frac{\pi}{2}\right)}$

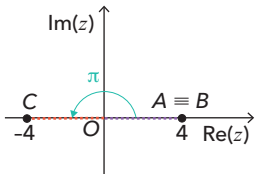
$-z = 3e^{i\left(\frac{\pi}{2}+\pi\right)} = 3e^{i\left(-\frac{\pi}{2}\right)}$



d) $z = 4 = 4e^{i0}$

$\overline{z} = 4e^{i0}$

$-z = 4e^{i\pi}$



45.

a) $\cos\theta - i\,\text{sen}\,\theta = \overline{\cos\theta + i\,\text{sen}\,\theta} = e^{i(-\theta)}$

b) $-\cos\theta - i\,\text{sen}\,\theta = -(\cos\theta + i\,\text{sen}\,\theta) = -e^{i\theta} = e^{i(\pi+\theta)}$

c) $\text{sen}\,\theta + i\cos\theta = \cos\left(\frac{\pi}{2}-\theta\right) + i\,\text{sen}\left(\frac{\pi}{2}-\theta\right) = e^{i\left(\frac{\pi}{2}-\theta\right)}$

d) $\text{sen}\,\theta - i\cos\theta = \cos\left(-\frac{\pi}{2}+\theta\right) + i\,\text{sen}\left(-\frac{\pi}{2}+\theta\right) =$

$= e^{i\left(-\frac{\pi}{2}+\theta\right)}$

Outro processo

$\text{sen}\,\theta - i\cos\theta = \overline{\text{sen}\,\theta + i\cos\theta} = \overline{e^{i\left(\frac{\pi}{2}-\theta\right)}} = e^{i\left(-\frac{\pi}{2}+\theta\right)}$

↑ alínea c)

46.

a) $z_1 \times z_2 = 2e^{i\frac{\pi}{8}} \times 3e^{i\left(-\frac{\pi}{4}\right)} = 2 \times 3e^{i\left(\frac{\pi}{8}-\frac{\pi}{4}\right)} = 6e^{i\left(-\frac{\pi}{8}\right)}$

b) $z_1 \times \overline{z}_2 = 2e^{i\frac{\pi}{8}} \times 3e^{i\frac{\pi}{4}} = 2 \times 3e^{i\left(\frac{\pi}{8}+\frac{\pi}{4}\right)} = 6e^{i\frac{3\pi}{8}}$

c) $z_1 \times z_2 = 3e^{i\left(-\frac{\pi}{4}\right)} \times 3e^{i\left(-\frac{\pi}{4}\right)} = 3 \times 3e^{i\left(-\frac{\pi}{4}-\frac{\pi}{4}\right)} = 9e^{i\left(-\frac{\pi}{2}\right)}$

d) $z_1 \times z_3 = 2e^{i\frac{\pi}{8}} \times 5i = 2e^{i\frac{\pi}{8}} \times 5e^{i\frac{\pi}{2}} =$

$= 10e^{i\left(\frac{\pi}{8}+\frac{\pi}{2}\right)} = 10e^{i\frac{5\pi}{8}}$

e) $z_1 \times z_2 \times z_3 = 2e^{i\frac{\pi}{8}} \times 3e^{i\left(-\frac{\pi}{4}\right)} \times 5e^{i\frac{\pi}{2}} =$

$= 6e^{i\left(-\frac{\pi}{8}\right)} \times 5e^{i\frac{\pi}{2}} = 30e^{i\left(-\frac{\pi}{8}+\frac{\pi}{2}\right)} = 30e^{i\frac{3\pi}{8}}$

47. $z \times \overline{z} = \rho e^{i\theta} \times \rho e^{-i\theta} =$

$= \rho \times \rho e^{i(\theta-\theta)} =$

$= \rho^2 e^{i0} =$

$= \rho^2$

48. Opção (D)

A multiplicação de um número complexo por $-i$ corresponde à rotação centrada na origem do vetor que é a sua imagem vetorial segundo um ângulo de amplitude -90°.

Assim, das opções apresentadas, o afixo de $-iz$ é z_4.

49.

a) $\frac{z_1}{z_2} = \frac{2e^{i\frac{3\pi}{5}}}{e^{i\left(-\frac{\pi}{10}\right)}} = 2e^{i\left(\frac{3\pi}{5}+\frac{\pi}{10}\right)} = 2e^{i\frac{7\pi}{10}}$

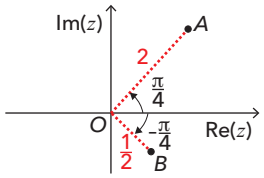
b) $\frac{z_1}{\overline{z}_2} = \frac{2e^{i\frac{3\pi}{5}}}{e^{i\frac{\pi}{10}}} = 2e^{i\left(\frac{3\pi}{5}-\frac{\pi}{10}\right)} = 2e^{i\frac{\pi}{2}}$

c) $\frac{3}{z_2} = \frac{3e^{i0}}{e^{i\left(-\frac{\pi}{10}\right)}} = 3e^{i\frac{\pi}{10}}$

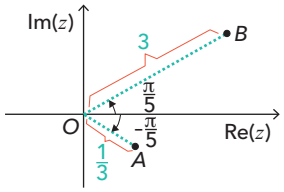
d) $\frac{-6i}{z_2} = \frac{6e^{i\left(-\frac{\pi}{2}\right)}}{e^{i\left(-\frac{\pi}{10}\right)}} = 6e^{i\left(-\frac{\pi}{2}+\frac{\pi}{10}\right)} = 6e^{i\left(-\frac{2\pi}{5}\right)}$

50. Sejam A o afixo de z e B o afixo do inverso de z.

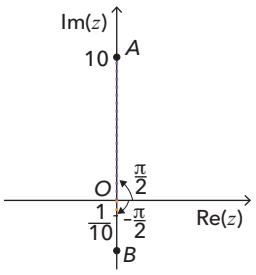
a) $\frac{1}{z} = \frac{e^{i0}}{2e^{i\frac{\pi}{4}}} = \frac{1}{2}e^{i\left(-\frac{\pi}{4}\right)}$



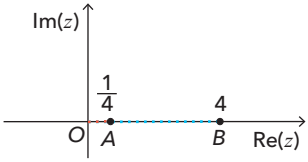
b) $\frac{1}{z} = \frac{e^{i0}}{\frac{1}{3}e^{i\left(-\frac{\pi}{5}\right)}} = 3e^{i\frac{\pi}{5}}$



c) $\frac{1}{z} = \frac{e^{i0}}{10e^{i\frac{\pi}{2}}} = \frac{1}{10}e^{i\left(-\frac{\pi}{2}\right)}$



d) $\frac{1}{z} = \frac{e^{i0}}{\frac{1}{4}e^{i0}} = 4e^{i0}$



51.

a) $\left(\cos\left(\frac{\pi}{7}\right) + i\,\text{sen}\left(\frac{\pi}{7}\right)\right)^3 = \left(e^{i\frac{\pi}{7}}\right)^3 = e^{i\frac{3\pi}{7}}$

b) $\left(3e^{i\frac{2\pi}{5}}\right)^5 = 3^5 e^{i\frac{10\pi}{5}} = 243e^{i0}$

c) $(-1-\sqrt{3}i)^4 = \left(2e^{i\frac{4\pi}{3}}\right)^4 = 2^4 e^{i\frac{16\pi}{3}} = 16e^{i\frac{4\pi}{3}}$

Cálculo auxiliar

$\rho = |-1-\sqrt{3}i| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2$

$\text{tg}\,\theta = \sqrt{3} \wedge \theta \in 3.^{\circ}Q$

Por exemplo, $\theta = \frac{4\pi}{3}$.

d) $\left(\frac{1+i}{1-i}\right)^8 = \left(\frac{\sqrt{2}e^{i\frac{\pi}{4}}}{\sqrt{2}e^{i\left(-\frac{\pi}{4}\right)}}\right)^8 = \left(e^{i\frac{\pi}{2}}\right)^8 = e^{i2\pi} = e^{i0}$

52.

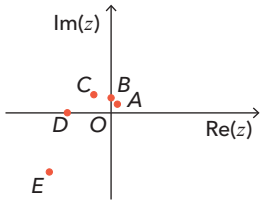
a) Seja $z = 2e^{i\frac{\pi}{4}}$ e A o seu afixo.

$z^2 = 2^2 e^{i\frac{\pi}{4}\times 2} = 4e^{i\frac{\pi}{2}}$ e seja B o seu afixo.

$z^3 = 2^3 e^{i\frac{\pi}{4}\times 3} = 8e^{i\frac{3\pi}{4}}$ e seja C o seu afixo.

$z^4 = 2^4 e^{i\frac{\pi}{4}\times 4} = 16e^{i\pi}$ e seja D o seu afixo.

$z^5 = 2^5 e^{i\frac{\pi}{4}\times 5} = 32e^{i\frac{5\pi}{4}}$ e seja E o seu afixo.



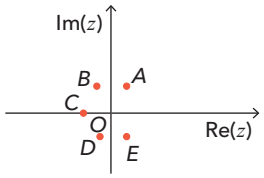
b) Seja $z = e^{i\frac{\pi}{3}}$ e A o seu afixo.

$z^2 = e^{i\frac{\pi}{3}\times 2} = e^{i\frac{2\pi}{3}}$ e seja B o seu afixo.

$z^3 = e^{i\frac{\pi}{3}\times 3} = e^{i\pi}$ e seja C o seu afixo.

$z^4 = e^{i\frac{\pi}{3}\times 4} = e^{i\frac{4\pi}{3}}$ e seja D o seu afixo.

$z^5 = e^{i\frac{\pi}{3}\times 5} = e^{i\frac{5\pi}{3}}$ e seja E o seu afixo.



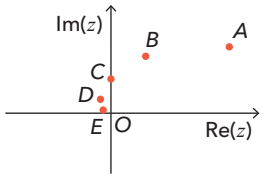
c) Seja $z = \frac{1}{2}e^{i\frac{\pi}{6}}$ e A o seu afixo.

$z^2 = \left(\frac{1}{2}\right)^2 e^{i\frac{\pi}{6}\times 2} = \frac{1}{4}e^{i\frac{\pi}{3}}$ e seja B o seu afixo.

$z^3 = \left(\frac{1}{2}\right)^3 e^{i\frac{\pi}{6}\times 3} = \frac{1}{8}e^{i\frac{\pi}{2}}$ e seja C o seu afixo.

$z^4 = \left(\frac{1}{2}\right)^4 e^{i\frac{\pi}{6}\times 4} = \frac{1}{16}e^{i\frac{2\pi}{3}}$ e seja D o seu afixo.

$z^5 = \left(\frac{1}{2}\right)^5 e^{i\frac{\pi}{6}\times 5} = \frac{1}{32}e^{i\frac{5\pi}{6}}$ e seja E o seu afixo.



53.

a) $z_1 \times z_2 = 2e^{i\frac{\pi}{5}} \times (-1+i) = 2e^{i\frac{\pi}{5}} \times \sqrt{2}e^{i\frac{3\pi}{4}} =$

$= 2\sqrt{2}e^{i\left(\frac{\pi}{5}+\frac{\pi}{4}\right)} = 2\sqrt{2}e^{i\frac{19\pi}{20}}$

b) $\frac{1}{z_2} = \frac{1}{\sqrt{2}e^{i\frac{3\pi}{4}}} = \frac{1}{\sqrt{2}}e^{i\left(-\frac{3\pi}{4}\right)} = \frac{\sqrt{2}}{2}e^{i\left(-\frac{3\pi}{4}\right)}$

c) $-z_1 = -2e^{i\frac{\pi}{5}} = 2e^{i\left(\pi+\frac{\pi}{5}\right)} = 2e^{i\frac{6\pi}{5}}$

d) $\frac{(z_1)^3}{z_2} = \frac{\left(2e^{i\frac{\pi}{5}}\right)^3}{\sqrt{2}e^{i\frac{3\pi}{4}}} = \frac{8e^{i\frac{3\pi}{5}}}{\sqrt{2}e^{i\frac{3\pi}{4}}}$

$= 4\sqrt{2}e^{i\left(\frac{3\pi}{5}-\frac{3\pi}{4}\right)} = 4\sqrt{2}e^{i\left(-\frac{3\pi}{20}\right)}$

e) $\bar{z}_1 \times \frac{1}{z_1} = 2e^{i\left(-\frac{\pi}{5}\right)} \times \frac{1}{2e^{i\frac{\pi}{5}}} = 2e^{i\left(-\frac{\pi}{5}\right)} \times \frac{1}{2}e^{i\left(-\frac{\pi}{5}\right)} = e^{i\left(-\frac{2\pi}{5}\right)}$

f) $(z_1)^5 + z_2 + i^{11} = \left(2e^{i\frac{\pi}{5}}\right)^5 - 1 + i + i^3 =$

$= 32e^{i\pi} - 1 + i - i =$

$= -32 - 1 =$

$= -33 =$

$= 33e^{i\pi}$

54.

a) $\frac{(3+i)^2}{2-i} - 2 = \frac{9+6i+i^2}{2-i} - 2 =$

$= \frac{(8+6i)(2+i)}{(2-i)(2+i)} - 2 =$

$= \frac{(16+8i+12i+6i^2}{2^2-i^2} - 2 =$

$= \frac{10+20i}{5} - 2 =$

$= 2 + 4i - 2 =$

$= 4i = 4e^{i\frac{\pi}{2}}$

b) $3i + \frac{(2-i)^3 - 8i^{43}}{1-i} = 3i + \frac{(2-i)^2(2-i) - 8 \times i^3}{1-i} =$

$= 3i + \frac{(4-4i+i^2)(2-i) + 8i}{1-i} =$

$= 3i + \frac{(3-4i)(2-i) + 8i}{1-i} =$

$= 3i + \frac{2-3i}{1-i} =$

$= 3i + \frac{(2-3i)(1+i)}{(1-i)(1+i)} =$

$= 3i + \frac{2+2i-3i-3i^2}{1^2-i^2} =$

$= 3i + \frac{5-i}{2} =$

$= \frac{5}{2} - \frac{1}{2}i + 3i =$

$= \frac{5}{2} + \frac{5}{2}i =$

$= \frac{5\sqrt{2}}{2} e^{i\frac{\pi}{4}}$

Cálculo auxiliar

$\rho = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \sqrt{\frac{25}{4} + \frac{25}{4}} = \frac{\sqrt{50}}{2} = \frac{5\sqrt{2}}{2}$

$\text{tg}\,\theta = \frac{\frac{5}{2}}{\frac{5}{2}} \wedge \theta \in 1.^{\circ}Q \Leftrightarrow \text{tg}\,\theta = 1 \wedge \theta \in 1.^{\circ}Q$

$\text{tg}\,\theta = \frac{\pi}{4}$, por exemplo.

55.

a) $\frac{\overline{z}}{z+1} = \frac{2e^{i\left(-\frac{4\pi}{3}\right)}}{2e^{i\frac{4\pi}{3}} + 1} =$

$= \frac{2e^{i\left(-\frac{4\pi}{3}\right)}}{2\left(\cos\left(\frac{4\pi}{3}\right) + i\,\text{sen}\left(\frac{4\pi}{3}\right)\right) + 1} =$

$= \frac{2e^{i\left(-\frac{4\pi}{3}\right)}}{2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + 1} =$

$= \frac{2e^{i\left(-\frac{4\pi}{3}\right)}}{-1 - \sqrt{3}i + 1} =$

$= \frac{2e^{i\left(-\frac{4\pi}{3}\right)}}{-\sqrt{3}i} =$

$= \frac{2e^{i\left(-\frac{4\pi}{3}\right)}}{\sqrt{3}e^{i\left(-\frac{\pi}{2}\right)}} =$

$= \frac{2\sqrt{3}}{3} e^{i\left(-\frac{4\pi}{3} + \frac{\pi}{2}\right)}$

$= \frac{2\sqrt{3}}{3} e^{i\left(-\frac{5\pi}{6}\right)}$

b) **i.** $w^2 = z \Leftrightarrow (\rho e^{i\theta})^2 = 2e^{i\frac{4\pi}{3}}$

$\Leftrightarrow \rho^2 e^{i2\theta} = 2e^{i\frac{4\pi}{3}}$

$\Leftrightarrow \rho^2 = 2 \wedge 2\theta = \frac{4\pi}{3} + 2k\pi, k \in \mathbb{Z}$

$\Leftrightarrow \underbrace{\rho = \sqrt{2}}_{r>0} \wedge \theta = \frac{2\pi}{3} + k\pi, k \in \mathbb{Z}$

ii. $\frac{w}{z} = \frac{\rho e^{i\theta}}{2e^{i\frac{4\pi}{3}}} = \frac{\rho}{2} e^{i\left(\theta - \frac{4\pi}{3}\right)}$

Para que $\frac{w}{z}$ seja um número real negativo, ρ tem de ser um número real maior do que 0 e $\theta - \frac{4\pi}{3} = \pi + 2k\pi, k \in \mathbb{Z} \Leftrightarrow \theta = \frac{7\pi}{3} + 2k\pi, k \in \mathbb{Z}$

56.

a) i é uma raiz quarta de 1, pois $i^4 = 1$.

b) $2 + i$ é uma raiz quadrada de $3 + 4i$, pois $(2+i)^2 = 3 + 4i$.

$(2+i)^2 = 4 + 4i + i^2 = 4 + 4i - 1 = 3 + 4i$

c) $1 + i$ é uma raiz de ordem 8 de 16, pois $(1+i)^8 = 16$.

$(1+i)^8 = ((1+i)^2)^4 = (1 + 2i + i^2)^4 = (2i)^4 = 2^4 i^4 = 16$

57. As raízes quadradas de $z = 2i$ são os números complexos $w = a + bi$ tais que $w^2 = z$, isto é:

$(a+bi)^2 = 2i \Leftrightarrow a^2 + 2abi + (bi)^2 = 2i$

$\Leftrightarrow a^2 + 2abi + b^2 i^2 = 2i$

$\Leftrightarrow (a^2 - b^2) + 2abi = 2i$

$\Leftrightarrow \begin{cases} a^2 - b^2 = 0 \\ 2ab = 2 \end{cases} \Leftrightarrow \begin{cases} a^2 = b^2 \\ ab = 1 \end{cases}$

$\Leftrightarrow \begin{cases} a = b \\ ab = 1 \end{cases} \vee \begin{cases} a = -b \\ ab = 1 \end{cases}$

$\Leftrightarrow \begin{cases} a = b \\ a^2 = 1 \end{cases} \vee \begin{cases} a = -b \\ \underbrace{-b^2 = 1}_{\text{condição impossível em } \mathbb{R}} \end{cases}$

$\Leftrightarrow \begin{cases} a = b \\ a^2 = 1 \end{cases}$

$\Leftrightarrow \begin{cases} a = b \\ a = 1 \end{cases} \vee \begin{cases} a = b \\ a = -1 \end{cases}$

$\Leftrightarrow \begin{cases} b = 1 \\ a = 1 \end{cases} \vee \begin{cases} b = -1 \\ a = -1 \end{cases}$

As raízes quadradas de $2i$ são os números complexos $1 + i$ e $-1 - i$.

58.

a) $z^4 = -1 \Leftrightarrow (\rho e^{i\theta}) = e^{i\pi}$

$\Leftrightarrow \rho^4 e^{i4\theta} = e^{i\pi}$

$\Leftrightarrow \rho^4 = 1 \wedge 4\theta = \pi + 2k\pi, k \in \mathbb{Z}$

$\Leftrightarrow \underbrace{\rho = 1}_{\rho > 0} \wedge \theta = \frac{\pi}{4} + \frac{k\pi}{2}, k \in \mathbb{Z}$

Se $k = 0$, então $z_0 = e^{i\frac{\pi}{4}}$.

Se $k = 1$, então $z_1 = e^{i\frac{3\pi}{4}}$.

Se $k = 2$, então $z_2 = e^{i\frac{5\pi}{4}}$.

Se $k = 3$, então $z_3 = e^{i\frac{7\pi}{4}}$.

C.S. $= \left\{e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}\right\}$

b) $z^5 = 32e^{i\frac{\pi}{6}} \Leftrightarrow (\rho e^{i\theta})^5 = 32e^{i\frac{\pi}{6}}$

$\Leftrightarrow \rho^5 e^{i5\theta} = 32e^{i\frac{\pi}{6}}$

$\Leftrightarrow \rho^5 = 32 \wedge 5\theta = \frac{\pi}{6} + 2k\pi, k \in \mathbb{Z}$

$\Leftrightarrow \rho = 2 \wedge \theta = \frac{\pi}{30} + \frac{2k\pi}{5}, k \in \mathbb{Z}$

Se $k = 0$, então $z_0 = 2e^{i\frac{\pi}{30}}$.

Se $k = 1$, então $z_1 = 2e^{i\frac{13\pi}{30}}$.

Se $k = 2$, então $z_2 = 2e^{i\frac{5\pi}{6}}$.

Se $k = 3$, então $z_3 = 2e^{i\frac{37\pi}{30}}$.

Se $k = 4$, então $z_4 = 2e^{i\frac{49\pi}{30}}$.

C.S. $= \left\{2e^{i\frac{\pi}{30}}, 2e^{i\frac{13\pi}{30}}, 2e^{i\frac{5\pi}{6}}, 2e^{i\frac{37\pi}{30}}, 2e^{i\frac{49\pi}{30}}\right\}$

c) $z^3 = -2\sqrt{3} + 6i$

Cálculo auxiliar

Seja $w = -2\sqrt{3} + 6i$.

$\rho = \sqrt{(-2\sqrt{3})^2 + 6^2} = \sqrt{48}$

$\text{tg}\,\theta = \frac{6}{-2\sqrt{3}} \wedge \theta \in 2.^{\circ}\,Q \Leftrightarrow \text{tg}\,\theta = -\sqrt{3} \wedge \theta \in 2.^{\circ}\,Q$

$\theta = \frac{2\pi}{3}$, por exemplo.

$\Leftrightarrow (\rho e^{i\theta})^3 = \sqrt{48}e^{i\frac{2\pi}{3}}$

$\Leftrightarrow \rho^3 e^{i3\theta} = \sqrt{48}e^{i\frac{2\pi}{3}}$

$\Leftrightarrow \rho^3 = \sqrt{48} \wedge 3\theta = \frac{2\pi}{3} + 2k\pi, k \in \mathbb{Z}$

$\Leftrightarrow \rho = \sqrt[3]{\sqrt{48}} \wedge \theta = \frac{2\pi}{9} + \frac{2k\pi}{3}, k \in \mathbb{Z}$

$\Leftrightarrow \rho = \sqrt[6]{48} \wedge \theta = \frac{2\pi}{9} + \frac{2k\pi}{3}, k \in \mathbb{Z}$

Se $k = 0$, então $z_0 = \sqrt[6]{48}e^{i\frac{2\pi}{9}}$.

Se $k = 1$, então $z_1 = \sqrt[6]{48}e^{i\frac{8\pi}{9}}$.

Se $k = 2$, então $z_2 = \sqrt[6]{48}e^{i\frac{14\pi}{9}}$.

C.S. $= \left\{\sqrt[6]{48}e^{i\frac{2\pi}{9}}, \sqrt[6]{48}e^{i\frac{8\pi}{9}}, \sqrt[6]{48}e^{i\frac{14\pi}{9}}\right\}$

59. Opção (B)

Os afixos das raízes índice 4 de w são vértices de um quadrado centrado na origem. Das opções apresentadas, apenas a (B) reúne estas condições.

60.

a) $\sqrt{3} - i = 2e^{-i\frac{\pi}{6}}$

Cálculo auxiliar

$\rho = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$

$\text{tg}\,\theta = \frac{-1}{\sqrt{3}} \wedge \theta \in 4.^{\circ}\,Q \Leftrightarrow \text{tg}\,\theta = -\frac{\sqrt{3}}{3} \wedge \theta \in 4.^{\circ}\,Q$

$\theta = -\frac{\pi}{6}$, por exemplo.

As raízes cúbicas de $2e^{i\left(-\frac{\pi}{6}\right)}$ são os números complexos z_k tais que:

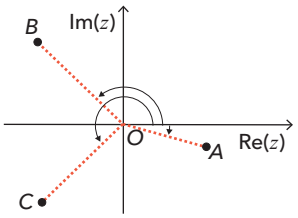
$z_k = \sqrt[3]{2}e^{i\left(\frac{-\frac{\pi}{6} + 2k\pi}{3}\right)}, k \in \{0, 1, 2\}$

Se $k = 0$, então $z_0 = \sqrt[3]{2}e^{i\left(-\frac{\pi}{18}\right)}$.

Se $k = 1$, então $z_1 = \sqrt[3]{2}e^{i\frac{11\pi}{18}}$.

Se $k = 2$, então $z_2 = \sqrt[3]{2}e^{i\frac{23\pi}{18}}$.

Sejam A, B e C os afixos dos números complexos z_0, z_1 e z_2, respetivamente.



$[ABC]$ é um triângulo equilátero centrado na origem.

b) $-81 = 81e^{i\pi}$

As raízes quartas de $81e^{i\pi}$ são os números complexos z_k tais que:

$z_k = \sqrt[4]{81}e^{i\left(\frac{\pi + 2k\pi}{4}\right)}, k \in \{0, 1, 2, 3\}$

$\Leftrightarrow z_k = 3e^{i\left(\frac{\pi + 2k\pi}{4}\right)}, k \in \{0, 1, 2, 3\}$

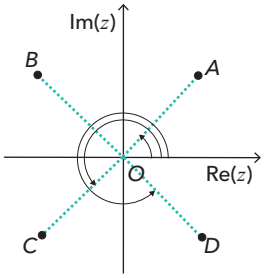
Se $k = 0$, então $z_0 = 3e^{i\frac{\pi}{4}}$.

Se $k = 1$, então $z_1 = 3e^{i\frac{3\pi}{4}}$.

Se $k = 2$, então $z_2 = 3e^{i\frac{5\pi}{4}}$.

Se $k = 3$, então $z_3 = 3e^{i\frac{7\pi}{4}}$.

Sejam A, B, C e D os afixos dos números complexos z_0, z_1, z_2 e z_3, respetivamente.



$[ABCD]$ é um quadrado centrado na origem.

c) $i = e^{i\frac{\pi}{2}}$

As raízes de ordem 6 de $e^{i\frac{\pi}{2}}$ são os números complexos z_k tais que:

$z_k = \sqrt[6]{1}e^{i\left(\frac{\frac{\pi}{2} + 2k\pi}{6}\right)}, k \in \{0, 1, 2, 3, 4, 5\}$

$\Leftrightarrow z_k = e^{i\left(\frac{\pi + 4k\pi}{12}\right)}, k \in \{0, 1, 2, 3, 4, 5\}$

Se $k = 0$, então $z_0 = e^{i\frac{\pi}{12}}$.

Se $k = 1$, então $z_1 = e^{i\frac{5\pi}{12}}$.

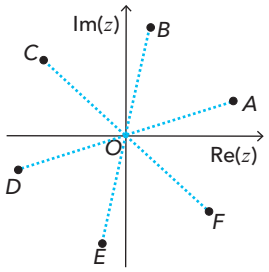
Se $k = 2$, então $z_2 = e^{i\frac{9\pi}{12}} = e^{i\frac{3\pi}{4}}$.

Se $k = 3$, então $z_3 = e^{i\frac{13\pi}{12}}$.

Se $k = 4$, então $z_4 = e^{i\frac{17\pi}{12}}$.

Se $k = 5$, então $z_5 = e^{i\frac{21\pi}{12}} = e^{i\frac{7\pi}{4}}$.

Sejam A, B, C, D, E e F os afixos dos números complexos z_0, z_1, z_2, z_3, z_4 e z_5, respetivamente.



[*ABCDEF*] é um hexágono regular centrado na origem.

61. Os módulos de qualquer raiz de ordem 6 de w são iguais. Neste caso, $\rho = 3$ e os argumentos das raízes correspondentes a valores de k consecutivos formam uma progressão aritmética de diferença $\frac{2\pi}{n}$.

Neste caso, $\frac{2\pi}{6} = \frac{\pi}{3}$.

Assim, se $z = 3e^{i\frac{2\pi}{3}}$ é uma raiz de ordem 6 de w, então as restantes cinco raízes são:

$z_1 = 3e^{i\left(\frac{2\pi}{3}+\frac{\pi}{3}\right)} = 3e^{i\pi}$

$z_2 = 3e^{i\left(\pi+\frac{\pi}{3}\right)} = 3e^{i\frac{4\pi}{3}}$

$z_3 = 3e^{i\left(\frac{4\pi}{3}+\frac{\pi}{3}\right)} = 3e^{i\frac{5\pi}{3}}$

$z_4 = 3e^{i\left(\frac{5\pi}{3}+\frac{\pi}{3}\right)} = 3e^{i2\pi}$

$z_5 = 3e^{i\left(2\pi+\frac{\pi}{3}\right)} = 3e^{i\frac{\pi}{3}}$

62. *A*, *B* e *C* são os afixos dos números complexos z_0, z_1 e z_2 que são raízes cúbicas de um determinado número complexo.

Assim:

$z_2 = 3e^{i\left(-\frac{\pi}{2}\right)}$; ponto $C(0, -3)$

$z_0 = 3e^{i\left(-\frac{\pi}{2}+\frac{2\pi}{3}\right)} = 3e^{i\frac{\pi}{6}} = 3\left(\cos\left(\frac{\pi}{6}\right) + i\,\text{sen}\left(\frac{\pi}{6}\right)\right) =$

$= 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$

$B\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$

$z_1 = 3e^{i\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)} = 3e^{i\frac{5\pi}{6}} = 3\left(\cos\left(\frac{5\pi}{6}\right) + i\,\text{sen}\left(\frac{5\pi}{6}\right)\right) =$

$= 3\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i$;

$A\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$

63. Como z_1 e z_2 são duas raízes consecutivas de um certo número complexo, então os seus argumentos diferem de $\frac{2\pi}{n}$.

Assim:

$\frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{2\pi}{n} \Leftrightarrow \frac{2\pi}{6} = \frac{2\pi}{n} \Leftrightarrow n = 6$

Como $z_1 = 2e^{i\left(\frac{\pi}{6}+\frac{2\pi}{6}\right)}$ e $z_2 = 2e^{i\frac{\pi}{6}}$, então as restantes raízes índice 6 de z são:

$z_3 = 2e^{i\left(\frac{\pi}{6}+\frac{2\pi}{6}\right)} = 2e^{i\frac{\pi}{2}}$

$z_4 = 2e^{i\left(\frac{\pi}{2}+\frac{2\pi}{6}\right)} = 2e^{i\frac{5\pi}{6}}$

$z_5 = 2e^{i\left(\frac{5\pi}{6}+\frac{2\pi}{6}\right)} = 2e^{i\frac{7\pi}{6}}$

$z_6 = 2e^{i\left(\frac{7\pi}{6}+\frac{2\pi}{6}\right)} = 2e^{i\frac{3\pi}{2}}$

64.

a) $z_6 + 2z^3 - 8 = 0$

$\Leftrightarrow z^3 = \frac{-2 \pm \sqrt{4+32}}{2}$

$\Leftrightarrow z^3 = 2 \vee z^3 = -4$

$\Leftrightarrow \rho^3 e^{i3\theta} = 2e^{i0} \vee \rho^3 e^{i3\theta} = 4e^{i\pi}$

$\Leftrightarrow (\rho = \sqrt[3]{2} \wedge 3\theta = 2k\pi) \vee (\rho = \sqrt[3]{4} \wedge 3\theta = \pi + 2k\pi)$, $k \in \mathbb{Z}$

$\Leftrightarrow \left(\rho = \sqrt[3]{2} \wedge \theta = \frac{2k\pi}{3}\right) \vee \left(\rho = \sqrt[3]{4} \wedge \theta = \frac{\pi}{3} + \frac{2k\pi}{3}\right)$, $k \in \mathbb{Z}$

Se $k = 0$, então $z = \sqrt[3]{2}e^{i0} \vee z = \sqrt[3]{4}e^{i\frac{\pi}{3}}$.

Se $k = 1$, então $z = \sqrt[3]{2}e^{i\frac{2\pi}{3}} \vee z = \sqrt[3]{4}e^{i\pi}$.

Se $k = 2$, então $z = \sqrt[3]{2}e^{i\frac{4\pi}{3}} \vee z = \sqrt[3]{4}e^{i\frac{5\pi}{3}}$.

C.S. $= \left\{\sqrt[3]{2}e^{i0}, \sqrt[3]{2}e^{i\frac{2\pi}{3}}, \sqrt[3]{2}e^{i\frac{4\pi}{3}}, \sqrt[3]{4}e^{i\frac{\pi}{3}}, \sqrt[3]{4}e^{i\pi}, \sqrt[3]{4}e^{i\frac{5\pi}{3}}\right\}$

b) $z^3 = i^{14} + i^{15} + i^{16} \Leftrightarrow z^3 = i^2 + i^3 + i^0$

$\Leftrightarrow z^3 = -1 - i + 1$

$\Leftrightarrow z^3 = -i$

$\Leftrightarrow \rho^3 e^{i3\theta} = e^{-i\frac{\pi}{2}}$

$\Leftrightarrow \rho^3 = 1 \wedge 3\theta = -\frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}$

$\Leftrightarrow \rho = 1 \wedge \theta = -\frac{\pi}{6} + \frac{2k\pi}{3}, k \in \mathbb{Z}$

Se $k = 0$, então $z = e^{i\left(-\frac{\pi}{6}\right)}$.

Se $k = 1$, então $z = e^{i\frac{\pi}{2}}$.

Se $k = 2$, então $z = e^{i\frac{7\pi}{6}}$.

C.S. $= \left\{e^{i\left(-\frac{\pi}{6}\right)}, e^{i\frac{\pi}{2}}, e^{i\frac{7\pi}{6}}\right\}$

c) $(1+i)z^5 = 3i$

$\Leftrightarrow z^5 = \frac{3i}{1+i}$

$\Leftrightarrow z^5 = \frac{3i}{1+i} \times \frac{1-i}{1-i}$

$\Leftrightarrow z^5 = \frac{3i - 3i^2}{1+1}$

$\Leftrightarrow z^5 = \frac{3}{2} + \frac{3}{2}i$

Cálculo auxiliar

$\rho = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{\frac{9}{4}+\frac{9}{4}} = \frac{\sqrt{18}}{2} = \frac{3\sqrt{2}}{2}$

$\text{tg}\,\theta = \frac{\frac{3}{2}}{\frac{3}{2}} \wedge \theta \in 1.^{\circ}\text{Q} \Leftrightarrow \text{tg}\,\theta = 1 \wedge \theta \in 1.^{\circ}\text{Q}$

$\text{tg}\,\theta = \frac{\pi}{4}$, por exemplo.

$\Leftrightarrow \rho^5 e^{i5\theta} = \frac{3\sqrt{2}}{2} e^{i\frac{\pi}{4}}$

$\Leftrightarrow \rho^5 = \frac{3\sqrt{2}}{2} \wedge 5\theta = \frac{\pi}{4} + 2k\pi, k \in \mathbb{Z}$

$\Leftrightarrow \rho^5 = \sqrt[5]{\frac{3\sqrt{2}}{2}} \wedge \theta = \frac{\pi}{20} + \frac{2k\pi}{5}, k \in \mathbb{Z}$

Se $k = 0$, então $z = \sqrt[5]{\frac{3\sqrt{2}}{2}}\, e^{i\frac{\pi}{20}}$.

Se $k = 1$, então $z = \sqrt[5]{\frac{3\sqrt{2}}{2}}\, e^{i\frac{9\pi}{20}}$.

Se $k = 2$, então $z = \sqrt[5]{\frac{3\sqrt{2}}{2}}\, e^{i\frac{17\pi}{20}}$.

Se $k = 3$, então $z = \sqrt[5]{\frac{3\sqrt{2}}{2}}\, e^{i\frac{5\pi}{4}}$.

Se $k = 4$, então $z = \sqrt[5]{\frac{3\sqrt{2}}{2}}\, e^{i\frac{33\pi}{20}}$.

C.S. $= \left\{\sqrt[5]{\frac{3\sqrt{2}}{2}}\, e^{i\frac{\pi}{20}}, \sqrt[5]{\frac{3\sqrt{2}}{2}}\, e^{i\frac{9\pi}{20}}, \sqrt[5]{\frac{3\sqrt{2}}{2}}\, e^{i\frac{17\pi}{20}}, \sqrt[5]{\frac{3\sqrt{2}}{2}}\, e^{i\frac{5\pi}{4}}, \sqrt[5]{\frac{3\sqrt{2}}{2}}\, e^{i\frac{33\pi}{20}}\right\}$

d) $\bar{z}^3 - 3z = 0$

$\Leftrightarrow \bar{z}^3 = 3z$

$\Leftrightarrow (\rho e^{-i\theta})^3 = 3\rho e^{i\theta}$

$\Leftrightarrow \rho^3 e^{-i3\theta} = 3\rho e^{i\theta}$

$\Leftrightarrow \rho^3 = 3\rho \wedge -3\theta = \theta + 2k\pi,\ k \in \mathbb{Z}$

$\Leftrightarrow \rho(\rho^2 - 3) = 0 \wedge -4\theta = 2k\pi,\ k \in \mathbb{Z}$

$\Leftrightarrow \left(\underbrace{\rho = 0}_{\text{se } \rho = 0, \text{ então } z = 0} \wedge \theta = \frac{k\pi}{2}\right) \vee \left(\underbrace{\rho = \sqrt{3}}_{\rho > 0} \wedge \theta = \frac{k\pi}{2}\right),\ k \in \mathbb{Z}$

Se $k = 0$, então $z = \sqrt{3}e^{i0} = \sqrt{3}$.

Se $k = 1$, então $z = \sqrt{3}e^{i\frac{\pi}{2}} = \sqrt{3}i$.

Se $k = 2$, então $z = \sqrt{3}e^{i\pi} = -\sqrt{3}$.

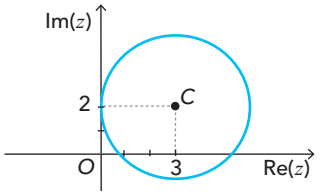
Se $k = 3$, então $z = \sqrt{3}e^{i\frac{3\pi}{2}} = -\sqrt{3}i$.

C.S. $= \{0, \sqrt{3}, \sqrt{3}i, -\sqrt{3}, -\sqrt{3}i\}$

65.

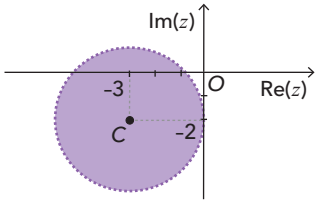
a) $|z - 3 - 2i| = 3 \Leftrightarrow |z - (3 + 2i)| = 3$

Representa a circunferência de centro no afixo de $z_1 = 3 + 2i$ e raio 3.



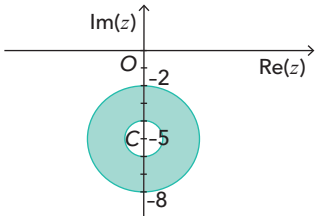
b) $|z + 3 + 2i| < 3 \Leftrightarrow |z - (-3 - 2i)| < 3$

Representa o interior do círculo de centro no afixo de $z_1 = -3 - 2i$ e raio 3.



c) $1 \leq |z + 5i| \leq 3 \Leftrightarrow |z - (-5i)| \geq 1 \wedge |z - (-5i)| \leq 3$

Representa a interseção do exterior da circunferência (incluindo a fronteira) de centro no afixo de $z_1 = -5i$ e raio 1, com o círculo com o mesmo centro e raio 3.



66. $z_1 = 1 + \sqrt{7}i$ raio = 3

$z_0 = 0$

$|z_1 - z_0| = |1 + \sqrt{7}i - 0| =$

$= \sqrt{1^2 + (\sqrt{7})^2} =$

$= \sqrt{1 + 7} =$

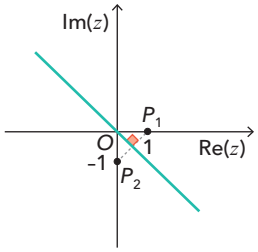
$= \sqrt{8}$

Como $|z_1 - z_0| < 3$, então o afixo de z_1 encontra-se no interior do círculo de centro na origem e raio 3.

67.

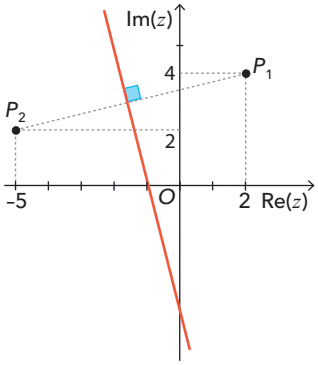
a) $|z + i| = |z - 1| \Leftrightarrow |z - (-i)| = |z - 1|$

Representa a mediatriz do segmento de reta $[P_1P_2]$ que tem como extremos os afixos de $z_1 = -i$ e $z_2 = 1$.



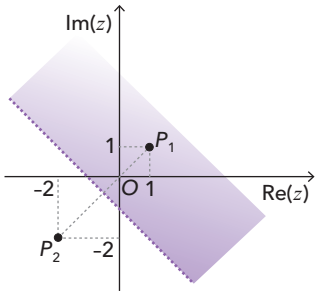
b) $|z - 2 - 4i| = |z + 5 - 2i| \Leftrightarrow |z - (2 + 4i)| = |z - (-5 + 2i)|$

Representa a mediatriz do segmento de reta $[P_1P_2]$ cujos extremos são os afixos de $z_1 = 2 + 4i$ e $z_2 = -5 + 2i$.



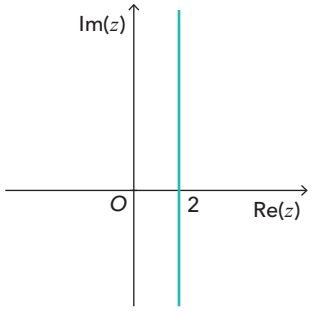
c) $|1 + i - z| < |z + 2 + 2i| \Leftrightarrow |z - (1 + i)| < |z - (-2 - 2i)|$

Representa o semiplano aberto definido pela mediatriz de $[P_1P_2]$ e que contém o ponto P_1, onde P_1 e P_2 são os afixos de $z_1 = 1 + i$ e $z_2 = -2 - 2i$, respetivamente.

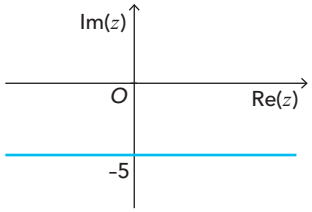


68.

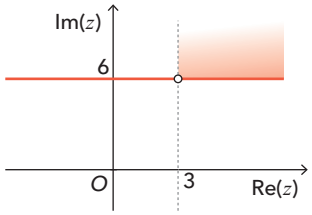
a) $\text{Re}(z) = 2$ representa a reta vertical que passa no ponto das coordenadas (2, 0).



b) $\text{Im}(z) = -5$ representa a reta horizontal que passa no ponto das coordenadas (0, –5).



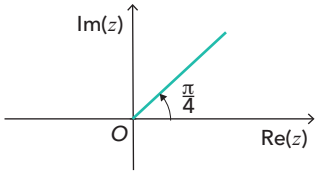
c) $\text{Re}(z) > 3 \wedge \text{Im}(z) \geq 6$



69.

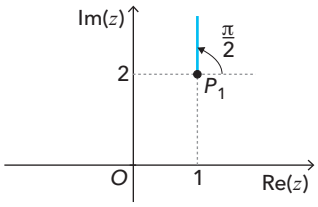
a) $\arg(z) = \frac{\pi}{4}$

Representa a semirreta com origem em O e que faz um ângulo de $\frac{\pi}{4}$ rad com o semieixo real positivo.



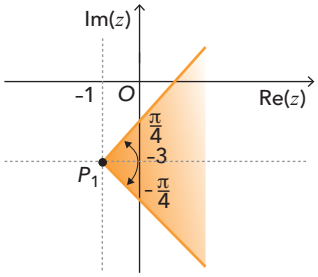
b) $\arg(z - 1 - 2i) = \frac{\pi}{2} \Leftrightarrow \arg(z - (1 + 2i)) = \frac{\pi}{2}$

Representa a semirreta com origem no afixo de $z_1 = 1 + 2i$ e que faz um ângulo de $\frac{\pi}{2}$ rad com o semieixo real positivo.



c) $-\frac{\pi}{4} \leq \arg(z + 1 + 3i) \leq \frac{\pi}{4} \Leftrightarrow -\frac{\pi}{4} \leq \arg(z - (-1 - 3i)) \leq \frac{\pi}{4}$

Representa o ângulo do vértice no afixo de $z_1 = -1 - 3i$ (P_1) cujos lados origem e extremidade são as semirretas de origem em P_1 e que fazem um ângulo de $-\frac{\pi}{4}$ e $\frac{\pi}{4}$ rad, respetivamente, com o semieixo real positivo.

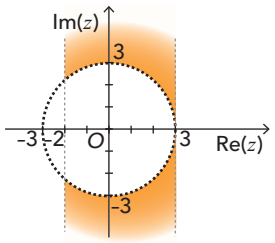


70.

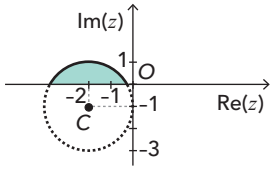
a) $-2 < \text{Re}(z) < 3 \wedge \left|\frac{1}{z}\right| < \frac{1}{3}$

Cálculo auxiliar

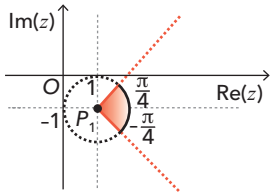
$\left|\frac{1}{z}\right| < \frac{1}{3} \Leftrightarrow \frac{1}{|z|} < \frac{1}{3} \Leftrightarrow |z| > 3$



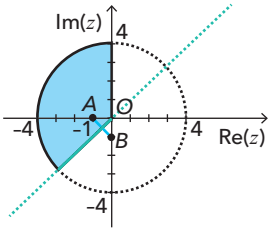
b) $|z + 2 + i| \wedge \text{Im}(z) \geq 0$



c) $-\frac{\pi}{4} \leq \arg(z - 1 + i) \leq \frac{\pi}{4} \wedge |z - 1 + i| \leq 1$



71. $|z| \leq 4 \wedge \text{Re}(z) \leq 0 \wedge |z+1| \leq |z+i|$



Observe-se que a condição em variável complexa $|z+1| = |z+i|$ corresponde, em $\mathbb{R}^2$, à bissetriz dos quadrantes ímpares.

Assim, a área a sombreado é $\frac{3}{8}$ da área do círculo de centro na origem e de raio 4.

$A = \frac{3}{8} \times \pi \times 4^2 = 6\pi$ u.a.

72.

a) $|z-(3-3i)| \leq 2 \wedge -\frac{\pi}{4} \leq \arg(z) \leq 0$

b) $\left|z-\left(\frac{7}{2}+3i\right)\right| = 3 \wedge \frac{\pi}{2} \leq \arg\left(z-\left(\frac{7}{2}+3i\right)\right) \leq \pi$

c) $0 \leq \arg(z-(2-4i)) \leq \frac{\pi}{4} \wedge \text{Re}(z) \leq 6$

d) $1 \leq |z-i| \leq 2 \wedge 1 \leq |z+i| \leq 2$

73. $\left|\frac{z-2+3i}{1+2i-z}\right| = 1 \Leftrightarrow |z-2+3i| = |1+2i-z|$

$\Leftrightarrow |z-(2-3i)| = |z-(1+2i)|$

Representa a mediatriz do segmento de reta $[AB]$, onde A é o afixo de $2-3i$ e B é o afixo de $1+2i$.

74. $2|z-1| = |z+2|$

Sendo $z = x+yi$:

$2|(x+yi)-1| = |(x+yi)+2|$

$\Leftrightarrow 2|(x-1)+yi| = |(x+2)+yi|$

$\Leftrightarrow 2\sqrt{(x-1)^2+y^2} = \sqrt{(x+2)^2+y^2}$

$\Leftrightarrow \left(\sqrt{(x-1)^2+y^2}\right)^2 = \left(\sqrt{(x+2)^2+y^2}\right)^2$

$\Leftrightarrow 4(x^2-2x+1+y^2) = x^2+4x+4+y^2$

$\Leftrightarrow 4x^2-8x+4+4y^2 = x^2+4x+4+y^2$

$\Leftrightarrow 3x^2+3y^2-12x = 0$

$\Leftrightarrow x^2-4x+y^2 = 0$

$\Leftrightarrow x^2-4x+2^2+y^2 = 2^2$

$\Leftrightarrow (x-2)^2+y^2 = 4$

Representa uma circunferência de centro no ponto de coordenadas (2, 0) e raio 2.

75.

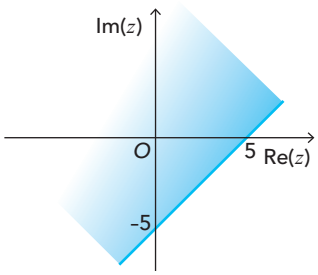
a) $\text{Re}(z) - \text{Im}(z) \leq 5$

Sendo $z = x+yi$:

$\text{Re}(x+yi) - \text{Im}(x+yi) \leq 5 \Leftrightarrow x-y \leq 5$

$\Leftrightarrow -y \leq -x+5$

$\Leftrightarrow y \geq x-5$



b) $z \times \bar{z} \leq 4\,\text{Re}(z)$

Sendo $z = x+yi$:

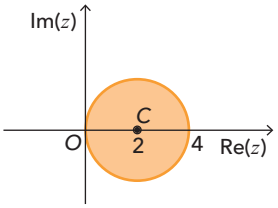
$(x+yi) \times (x-yi) \leq 4\,\text{Re}(x+yi)$

$\Leftrightarrow x^2-(yi)^2 \leq 4x$

$\Leftrightarrow x^2+y^2 \leq 4x$

$\Leftrightarrow x^2-4x+2^2+y^2 \leq 2^2$

$\Leftrightarrow (x-2)^2+y^2 \leq 4$



c) $z \times \bar{z} = -z-\bar{z}$

Sendo $z = x+yi$:

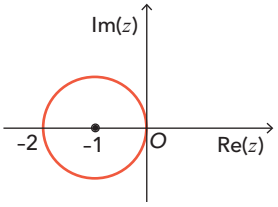
$(x+yi) \times (x-yi) = -(x+yi) - (x-yi)$

$\Leftrightarrow x^2-(yi)^2 = -x-yi-x+yi$

$\Leftrightarrow x^2+y^2 = -2x$

$\Leftrightarrow x^2+2x+1+y^2 = 1$

$\Leftrightarrow (x+1)^2+y^2 = 1$



d) $\operatorname{Im}\left(\frac{1}{z+2}\right) \geq 1$

Sendo $z = x + yi$:

$\operatorname{Im}\left(\frac{1}{x+yi+2}\right) \geq 1$

$\Leftrightarrow \operatorname{Im}\left(\frac{1}{(x+2)+yi}\right) \geq 1$

$\Leftrightarrow \operatorname{Im}\left(\frac{(x+2)-yi}{((x+2)+yi)((x+2)-yi)}\right) \geq 1$

$\Leftrightarrow \operatorname{Im}\left(\frac{(x+2)-yi}{(x+2)^2-(yi)^2}\right) \geq 1$

$\Leftrightarrow \operatorname{Im}\left(\frac{x+2}{(x+2)^2+y^2} - \frac{y}{(x+2)^2+y^2}i\right) \geq 1$

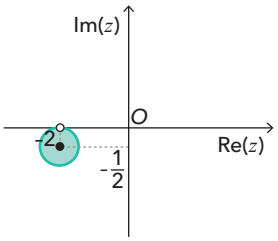
$\Leftrightarrow -\frac{y}{(x+2)^2+y^2} \geq 1$

$\Leftrightarrow -y \geq (x+2)^2 + y^2 \wedge (x, y) \neq (-2, 0)$

$\Leftrightarrow (x+2)^2 + y^2 + y \leq 0 \wedge (x, y) \neq (-2, 0)$

$\Leftrightarrow (x+2)^2 + y^2 + y + \left(\frac{1}{2}\right)^2 \leq \left(\frac{1}{2}\right)^2 \wedge (x, y) \neq (-2, 0)$

$\Leftrightarrow (x+2)^2 + \left(y + \frac{1}{2}\right)^2 \leq \frac{1}{4}$



Aprende fazendo (2) – páginas 68 e 69

76. Opção (B)

Tem-se que:

$i^3 = -i$

$i^6 = i^2 = -1$

$e^{i\frac{\pi}{2}} = i$

$e^{i\frac{3\pi}{2}} = -i$

Assim, o número complexo $i^6 = -1$ não é um imaginário puro, mas sim um número real.

77.

a) $e^{i\frac{\pi}{6}} = \cos\left(\frac{\pi}{6}\right) + i\operatorname{sen}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

b) $\sqrt{2}e^{i\left(-\frac{\pi}{4}\right)} = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\operatorname{sen}\left(-\frac{\pi}{4}\right)\right) =$

$= \sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) =$

$= 1 - i$

c) $2e^{i\frac{5\pi}{3}} = 2\left(\cos\left(\frac{5\pi}{3}\right) + i\operatorname{sen}\left(\frac{5\pi}{3}\right)\right) =$

$= 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) =$

$= 1 - \sqrt{3}i$

d) $4e^{i\frac{\pi}{2}} = 4\left(\cos\left(\frac{\pi}{2}\right) + i\operatorname{sen}\left(\frac{\pi}{2}\right)\right) =$

$= 4(0 + i) =$

$= 4i$

e) $\sqrt{5}e^{i\pi} = \sqrt{5}(\cos\pi + i\operatorname{sen}\pi) = \sqrt{5}(-1 + 0i) = -\sqrt{5}$

f) $\frac{1}{6}e^{i\frac{7\pi}{2}} = \frac{1}{6}\left(\cos\left(\frac{7\pi}{2}\right) + i\operatorname{sen}\left(\frac{7\pi}{2}\right)\right) =$

$= \frac{1}{6}(0 - i) =$

$= -\frac{1}{6}i$

g) $e^{i0} = \cos 0 + i\operatorname{sen} 0 = 1 + 0i = 1$

78.

a) $i = e^{i\frac{\pi}{2}}$

b) $1 = e^{i0}$

c) $-\sqrt{11} = \sqrt{11}e^{i\pi}$

d) $-4i = 4e^{i\left(-\frac{\pi}{2}\right)}$

e) $1 - i = \sqrt{2}e^{i\left(-\frac{\pi}{4}\right)}$

Cálculo auxiliar

$\rho = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\operatorname{tg}\theta = -1 \wedge \theta \in 4.^{\circ}\,Q$

Por exemplo, $\theta = -\frac{\pi}{4}$.

f) $\frac{1}{4} + \frac{1}{4}i = \frac{\sqrt{2}}{4}e^{i\frac{\pi}{4}}$

Cálculo auxiliar

$\rho = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2} = \sqrt{\frac{1}{16} + \frac{1}{16}} = \sqrt{\frac{2}{16}} = \frac{\sqrt{2}}{4}$

$\operatorname{tg}\theta = 1 \wedge \theta \in 1.^{\circ}\,Q$

Por exemplo, $\theta = \frac{\pi}{4}$.

g) $1 - \sqrt{3}i = 2e^{i\left(-\frac{\pi}{3}\right)}$

Cálculo auxiliar

$\rho = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2$

$\operatorname{tg}\theta = -\sqrt{3} \wedge \theta \in 4.^{\circ}\,Q$

Por exemplo, $\theta = -\frac{\pi}{3}$.

h) $-\sqrt{2} - \sqrt{6}i = 2\sqrt{2}e^{i\frac{4\pi}{3}}$

Cálculo auxiliar

$\rho = \sqrt{(-\sqrt{2})^2 + (-\sqrt{6})^2} = \sqrt{2+6} = \sqrt{8} = 2\sqrt{2}$

$\operatorname{tg}\theta = \sqrt{3} \wedge \theta \in 3.^{\circ}\,Q$

Por exemplo, $\theta = \frac{4\pi}{3}$.

i) $-\sqrt{3} + i = 2e^{i\frac{5\pi}{6}}$

Cálculo auxiliar

$\rho = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{3+1} = 2$

$\operatorname{tg}\theta = -\frac{\sqrt{3}}{3} \wedge \theta \in 2.^{\circ}\,Q$

Por exemplo, $\theta = \frac{5\pi}{6}$.

79.

a) $z = 3e^{i\frac{\pi}{7}}$

$\overline{z} = 3e^{i\left(-\frac{\pi}{7}\right)}$

$-z = 3e^{i\left(\frac{\pi}{7}+\pi\right)} = 3e^{i\frac{8\pi}{7}}$

$\frac{1}{z} = \frac{1}{3}e^{i\left(-\frac{\pi}{7}\right)}$

b) $z = \sqrt{2}e^{i\left(-\frac{\pi}{3}\right)}$

$\overline{z} = \sqrt{2}e^{i\frac{\pi}{3}}$

$-z = \sqrt{2}e^{i\left(-\frac{\pi}{3}+\pi\right)} = \sqrt{2}e^{i\frac{2\pi}{3}}$

$\frac{1}{z} = \frac{1}{\sqrt{2}}e^{i\frac{\pi}{3}} = \frac{\sqrt{2}}{2}e^{i\frac{\pi}{3}}$

c) $z = 3i = 3e^{i\frac{\pi}{2}}$

$\overline{z} = 3e^{i\left(-\frac{\pi}{2}\right)}$

$-z = 3e^{i\left(\frac{\pi}{2}+\pi\right)} = 3e^{i\frac{3\pi}{2}}$

$\frac{1}{z} = \frac{1}{3}e^{i\left(-\frac{\pi}{2}\right)}$

d) $z = (-1-i)^2 = 1 + 2i + i^2 = 2i = 2e^{i\frac{\pi}{2}}$

$\overline{z} = 2e^{i\left(-\frac{\pi}{2}\right)}$

$-z = 2e^{i\left(\frac{\pi}{2}+\pi\right)} = 2e^{i\frac{3\pi}{2}}$

$\frac{1}{z} = \frac{1}{2}e^{i\left(-\frac{\pi}{2}\right)}$

80. $w = -1 = e^{i\pi}$

As raízes cúbicas de w são os valores de w_k tais que:

$w_k = \sqrt[3]{1}\,e^{i\left(\frac{\pi+2k\pi}{3}\right)}$, $k \in \{0, 1, 2\}$

$\Leftrightarrow w_k = e^{i\left(\frac{\pi+2k\pi}{3}\right)}$, $k \in \{0, 1, 2\}$

Se $k = 0$, então $w_0 = e^{i\frac{\pi}{3}}$.

Se $k = 1$, então $w_1 = e^{i\pi}$.

Se $k = 2$, então $w_2 = e^{i\frac{5\pi}{3}}$.

81.

a) $z^2 - 4z + 5 = 0 \Leftrightarrow z = \frac{4 \pm \sqrt{(-4)^2 - 4\times 1\times 5}}{2\times 1}$

$\Leftrightarrow z = \frac{4 \pm \sqrt{-4}}{2}$

$\Leftrightarrow z = \frac{4+2i}{2} \vee \frac{4-2i}{2}$

$\Leftrightarrow z = 2 + i \vee z = 2 - i$

C.S. $= \{2 + i, 2 - i\}$

b) $z^3 + 2z = 0 \Leftrightarrow z(z^2 + 2) = 0$

$\Leftrightarrow z = 0 \vee z^2 = -2$

$\Leftrightarrow z = 0 \vee z = \sqrt{2}i \vee z = -\sqrt{2}i$

C.S. $= \{0, \sqrt{2}i, -\sqrt{2}i\}$

c) $z^6 + 64 = 0 \Leftrightarrow z^6 = -64$

As soluções desta equação são as raízes de ordem 6 do número complexo $-64 = 64e^{i\pi}$, ou seja, os valores z_k tais que:

$z_k = \sqrt[6]{64}\,e^{i\left(\frac{\pi+2k\pi}{6}\right)}$, $k \in \{0, 1, 2, 3, 4, 5\}$

$\Leftrightarrow z_k = 2e^{i\left(\frac{\pi+2k\pi}{6}\right)}$, $k \in \{0, 1, 2, 3, 4, 5\}$

Se $k = 0$, então $z_0 = 2e^{i\frac{\pi}{6}}$.

Se $k = 1$, então $z_1 = 2e^{i\frac{\pi}{2}}$.

Se $k = 2$, então $z_2 = 2e^{i\frac{5\pi}{6}}$.

Se $k = 3$, então $z_3 = 2e^{i\frac{7\pi}{6}}$.

Se $k = 4$, então $z_4 = 2e^{i\frac{3\pi}{2}}$.

Se $k = 5$, então $z_5 = 2e^{i\frac{11\pi}{6}}$.

C.S. $= \left\{2e^{i\frac{\pi}{6}}, 2e^{i\frac{\pi}{2}}, 2e^{i\frac{5\pi}{6}}, 2e^{i\frac{7\pi}{6}}, 2e^{i\frac{3\pi}{2}}, 2e^{i\frac{11\pi}{6}}\right\}$

82. Opção (A)

$|z + 2| = 3$ define uma circunferência no plano complexo de centro no afixo de $z_1 = -2$ e de raio 3.

83. Opção (D)

Seja $z = \rho e^{i\theta}$. Então, $2z = 2\rho e^{i\theta}$. Logo, $\arg(2z) = \theta$.

84.

a) $z = \sqrt{3} - \sqrt{3}i = \sqrt{6}e^{i\left(-\frac{\pi}{4}\right)}$

Cálculo auxiliar

$\rho = \sqrt{(\sqrt{3})^2 + (-\sqrt{3})^2} = \sqrt{3+3} = \sqrt{6}$

$\text{tg}\,\theta = \frac{-\sqrt{3}}{\sqrt{3}} \wedge \theta \in 4.^{\circ}\,Q \Leftrightarrow \text{tg}\,\theta = -1 \wedge \theta \in 4.^{\circ}\,Q$

Por exemplo, $\theta = -\frac{\pi}{4}$.

b) $w = -\frac{\sqrt{3}}{2} - \frac{1}{2}i = e^{i\frac{7\pi}{6}}$

Cálculo auxiliar

$\rho = \sqrt{\left(\frac{-\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}+\frac{1}{4}} = \sqrt{\frac{4}{4}} = 1$

$\text{tg}\,\theta = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} \wedge \theta \in 3.^{\circ}\,Q \Leftrightarrow \text{tg}\,\theta = \frac{1}{\sqrt{3}} \wedge \theta \in 3.^{\circ}\,Q \Leftrightarrow \theta = \frac{\pi}{6} + \pi$

Por exemplo, $\theta = \frac{7\pi}{6}$.

c) $z \times w = \sqrt{6}e^{i\left(-\frac{\pi}{4}\right)} \times e^{i\frac{7\pi}{6}} = \sqrt{6}e^{i\left(-\frac{\pi}{4}+\frac{7\pi}{6}\right)} = \sqrt{6}e^{i\frac{11\pi}{12}}$

d) $\frac{z}{w} = \frac{\sqrt{6}e^{i\left(-\frac{\pi}{4}\right)}}{e^{i\frac{7\pi}{6}}} = \sqrt{6}e^{i\left(-\frac{\pi}{4}-\frac{7\pi}{6}\right)} = \sqrt{6}e^{i\left(-\frac{17\pi}{12}\right)}$

e) $z^{-1} = \frac{1}{z} = \frac{1}{\sqrt{6}e^{i\left(-\frac{\pi}{4}\right)}} = \frac{\sqrt{6}}{6}e^{i\frac{\pi}{4}}$

f) $z^4 + 1 = \left(\sqrt{6}e^{i\left(-\frac{\pi}{4}\right)}\right)^4 + 1 = (\sqrt{6})^4 e^{i\left(-\frac{\pi}{4}\times 4\right)} + 1 =$

$= 36e^{i(-\pi)} + 1 = -36 + 1 = -35 = 35e^{i\pi}$

g) $\frac{\overline{w}}{i^{15}} = \frac{e^{i\left(-\frac{7\pi}{6}\right)}}{i^3} = \frac{e^{i\left(-\frac{7\pi}{6}\right)}}{-i} = \frac{e^{i\left(-\frac{7\pi}{6}\right)}}{e^{i\left(-\frac{\pi}{2}\right)}} = e^{i\left(-\frac{7\pi}{6}+\frac{\pi}{2}\right)} = e^{i\left(-\frac{2\pi}{3}\right)}$

85. $w = 1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$

As raízes quartas de $w = \sqrt{2}e^{i\frac{\pi}{4}}$ são os valores w_k tais que:

$w_k = \sqrt[4]{\sqrt{2}}e^{i\left(\frac{\frac{\pi}{4}+2k\pi}{3}\right)}$, $k \in \{0, 1, 2, 3\}$

$\Leftrightarrow w_k = \sqrt[8]{2}e^{i\left(\frac{\pi}{16}+\frac{2k\pi}{3}\right)}$, $k \in \{0, 1, 2, 3\}$

Se $k = 0$, então $z_0 = \sqrt[8]{2}e^{i\frac{\pi}{16}}$.

Se $k = 1$, então $z_1 = \sqrt[8]{2}e^{i\frac{9\pi}{16}}$.

Se $k = 2$, então $z_2 = \sqrt[8]{2}e^{i\frac{17\pi}{16}}$.

Se $k = 3$, então $z_3 = \sqrt[8]{2}e^{i\frac{25\pi}{16}}$.

86. Opção (B)

A região a sombreado contém os pontos do plano complexo que pertencem à interseção dos dois círculos representados e cujo coeficiente da parte imaginária é negativo ou nulo.

$|z| \leq 1 \wedge |z - 1| \leq 1 \wedge \text{Im}(z) \leq 0$

87. Opção (B)

$w^n + \frac{1}{w^n} =$

$= (e^{i\theta})^n + \frac{1}{(e^{i\theta})^n} =$

$= e^{i\theta n} + \frac{1}{e^{i\theta n}} =$

$= e^{i\theta n} + e^{i(-\theta n)} =$

$= \cos(n\theta) + i\,\text{sen}(n\theta) + \cos(-n\theta) + i\,\text{sen}(-n\theta) =$

$= \cos(n\theta) + i\,\text{sen}(n\theta) + \cos(n\theta) - i\,\text{sen}(n\theta) =$

$= 2\cos(n\theta)$

88. Seja $z = \sqrt{12} - 2i$.

Cálculo auxiliar

$\rho = \sqrt{(\sqrt{12})^2 + (-2)^2} = \sqrt{16} = 4$

$\text{tg}\,\theta = \frac{-2}{\sqrt{12}} \wedge \theta \in 4.^{\circ}\text{Q} \Leftrightarrow \text{tg}\,\theta = -\frac{2}{2\sqrt{3}} \wedge \theta \in 4.^{\circ}\text{Q}$

$\Leftrightarrow \text{tg}\,\theta = -\frac{\sqrt{3}}{3} \wedge \theta \in 4.^{\circ}\text{Q}$

Por exemplo, $\theta = -\frac{\pi}{6}$.

$z = 4e^{i\left(-\frac{\pi}{6}\right)}$

$z^n = \left(4e^{i\left(-\frac{\pi}{6}\right)}\right)^n = 4^n e^{i\left(-\frac{n\pi}{6}\right)}$

Para z^n ser um número real positivo, então $\arg(z^n) = 2k\pi$, $k \in \mathbb{Z}$, ou seja:

$-n\frac{\pi}{6} = 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow -n\pi = 12k\pi,\ k \in \mathbb{Z}$

$\Leftrightarrow n = -12k,\ k \in \mathbb{Z}$

Se $k = 0$, $n = 0 \in \mathbb{N}$

Se $k = 1$, $n = -12 \notin \mathbb{N}$

Se $k = -1$, $n = 12$, 12 é o menor número natural positivo tal que z^n é um número real positivo.

89.

a) Representação de $\cos\theta - i\,\text{sen}\,\theta$ na forma trigonométrica:

$\cos\theta - i\,\text{sen}\,\theta = \cos(-\theta) + i\,\text{sen}(-\theta) = e^{i(-\theta)}$

Representação de $\text{sen}\,\theta + i\cos\theta$ na forma trigonométrica:

$\text{sen}\,\theta + i\cos\theta = \cos\left(\frac{\pi}{2} - \theta\right) + i\,\text{sen}\left(\frac{\pi}{2} - \theta\right) = e^{i\left(\frac{\pi}{2} - \theta\right)}$

Assim:

$\left(\frac{\cos\theta - i\,\text{sen}\,\theta}{\text{sen}\,\theta + i\cos\theta}\right)^9 = \frac{e^{i(-\theta)}}{e^{i\left(\frac{\pi}{2} - \theta\right)}} =$

$= \left(e^{i\left(-\theta - \frac{\pi}{2} + \theta\right)}\right)^9 =$

$= \left(e^{i\left(-\frac{\pi}{2}\right)}\right)^9 =$

$= e^{i\left(-\frac{9\pi}{2}\right)} =$

$= \cos\left(-\frac{9\pi}{2}\right) + i\,\text{sen}\left(-\frac{9\pi}{2}\right) =$

$= 0 + i \times (-1) =$

$= -i$

b) $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^{2028} - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^{2026} =$

$= \left(e^{i\frac{\pi}{4}}\right)^{2028} - \left(e^{i\frac{\pi}{4}}\right)^{2026} =$

$= e^{i\frac{2028\pi}{4}} - e^{i\frac{2026\pi}{4}} =$

$= e^{i507\pi} - e^{i\frac{\pi}{2}} =$

$= -1 - i$

Aprende Fazendo Global – páginas 81 a 85

1. Opção (A)

Se $\arg(z) = -\frac{\pi}{7}$, então $\arg(\bar{z}) = -\left(-\frac{\pi}{7}\right) = \frac{\pi}{7}$.

2.

a) $\frac{z_1 + i^{23} + 5}{2 - i} = \frac{\sqrt{2}e^{i\frac{\pi}{4}} + i^3 + 5}{2 - i} =$

$= \frac{2\left(\cos\left(\frac{\pi}{4}\right) + i\,\text{sen}\left(\frac{\pi}{4}\right)\right) - i + 5}{2 - i} =$

$= \frac{1 + i - i + 5}{2 - i} =$

$= \frac{6}{2 - i} \times \frac{2 + i}{2 + i} =$

$= \frac{12 + 6i}{2^2 + 1^2} =$

$= \frac{12}{5} + \frac{6}{5}i$

b) $z_1 \times z_2 = \sqrt{2}\,e^{i\frac{\pi}{4}} \times 2e^{i\frac{5\pi}{6}} =$

$= 2\sqrt{2}e^{i\left(\frac{\pi}{4} + \frac{5\pi}{6}\right)} =$

$= 2\sqrt{2}e^{i\frac{13\pi}{12}}$

Cálculo auxiliar

$\rho_2 = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{3 + 1} = 2$

$\text{tg}\,\theta_2 = -\frac{\sqrt{3}}{2} \wedge \theta_2 \in 2.^{\circ}\text{Q}$

Por exemplo, $\theta_2 = \frac{5\pi}{6}$.

3.

a) $z = \frac{1}{2} - \frac{\sqrt{3}}{2}i = e^{i\left(-\frac{\pi}{3}\right)}$

Cálculo auxiliar

$\rho = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$

$\text{tg}\,\theta = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} \wedge \theta \in 4.^{\circ}\text{Q} \Leftrightarrow \text{tg}\,\theta = -\sqrt{3} \wedge \theta \in 4.^{\circ}\text{Q}$

Por exemplo, $\theta = -\frac{\pi}{3}$.

Se $z = e^{i\left(-\frac{\pi}{3}\right)}$ é uma raiz cúbica de w, então as restantes serão:

$z_1 = e^{i\left(-\frac{\pi}{3} + \frac{2\pi}{3}\right)} = e^{i\frac{\pi}{3}}$ e $z_2 = e^{i\left(\frac{\pi}{3} + \frac{2\pi}{3}\right)} = e^{i\pi}$

b) Como z é uma raiz cúbica de w, então $z^3 = w$.

Assim, $w = \left(e^{i\left(-\frac{\pi}{3}\right)}\right)^3 = e^{i(-\pi)} = -1$.

4. Opção (D)

$i^2 = -1$ e $(-1)^2 = 1$

$(-i)^2 = -1$ e $1^2 = 1$

$(2 + 2i)^2 = 4 + 8i + 4i^2 = 8i$ e $(2 - 2i)^2 = 4 - 8i + 4i^2 = -8i$

$(-2 + 2i)^2 = 4 - 8i + 4i^2 = -8i$ e $(2 - 2i)^2 = -8i$

Logo, $-2 + 2i$ e $2 - 2i$ são raízes quadradas do mesmo número complexo.

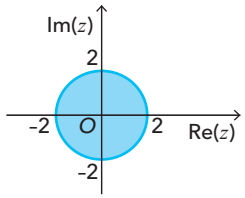
5.

a) $|z - 2| \geq 1 \wedge |z - 2| \leq 2 \Leftrightarrow 1 \leq |z - 2| \leq 2$

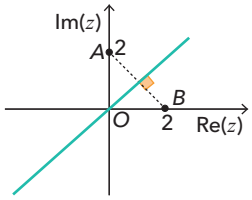
b) $|z - (-3 + i)| \leq |z - (2 - 2i)| \Leftrightarrow |z + 3 - i| \leq |z - 2 + 2i|$

6.

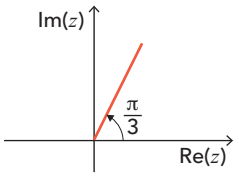
a) $|z| \leq 2$



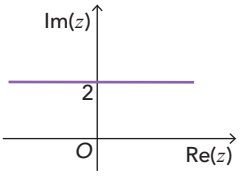
b) $|z - 2i| = |z - 2|$



c) $\arg(z) = \frac{\pi}{3}$



d) $\text{Im}(z) = 2$



7. Opção (A)

Seja $z = \rho e^{i\theta}$, com $0 < \theta < \frac{\pi}{2}$.

Então, $2z = 2\rho e^{i\theta}$ e $0 < \theta < \frac{\pi}{2}$.

8. Opção (B)

$\frac{i}{i} = i - i$

$\Leftrightarrow 1 = 0$, que é uma proposição falsa.

$\frac{i}{|i|} = i$

$\Leftrightarrow \frac{i}{1} = i$, que é uma proposição verdadeira.

$\arg(i) = 0$

$\Leftrightarrow \frac{\pi}{2} = 0$, que é uma proposição falsa.

$i^2 = i$

$\Leftrightarrow -1 = i$, que é uma proposição falsa.

9.

a) $z_1 = -2\sqrt{3} - 2i = 4e^{i\frac{7\pi}{6}}$

Cálculo auxiliar

$\rho_1 = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{12 + 4} = \sqrt{16} = 4$

$\text{tg}\,\theta = \frac{-2}{-2\sqrt{3}} \wedge \theta \in 3.^{\circ}Q \Leftrightarrow \text{tg}\,\theta = \frac{\sqrt{3}}{3} \wedge \theta \in 3.^{\circ}Q$

Por exemplo, $\theta = \frac{7\pi}{6}$.

Assim, $(z_1)^3 = \left(4e^{i\frac{7\pi}{6}}\right)^3 = 4^3 e^{i\left(3 \times \frac{7\pi}{6}\right)} = 64e^{i\frac{7\pi}{2}}$

b) $z_2 = \frac{1 - i}{1 + i} = \frac{(1 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{1 - 2i + i^2}{1^2 + 1^2} = \frac{-2i}{2} = -i = e^{i\left(-\frac{\pi}{2}\right)}$

As raízes quadradas de $z_2 = e^{i\left(-\frac{\pi}{2}\right)}$ são os valores de z_k tais que:

$z_k = \sqrt{1}\, e^{i\left(\frac{-\frac{\pi}{2} + 2k\pi}{2}\right)}, k \in \{0, 1\}$

$\Leftrightarrow z_k = e^{i\left(-\frac{\pi}{4} + k\pi\right)}, k \in \{0, 1\}$

Se $k = 0$, então $z_0 = e^{i\left(-\frac{\pi}{4}\right)} =$

$= \cos\left(-\frac{\pi}{4}\right) + i\,\text{sen}\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i.$

Se $k = 1$, então $z_1 = e^{i\frac{3\pi}{4}} =$

$= \cos\left(\frac{3\pi}{4}\right) + i\,\text{sen}\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i.$

c) $\frac{z_1}{z_3} = \frac{4e^{i\frac{7\pi}{6}}}{e^{i\left(\frac{\pi}{2} + \alpha\right)}} = 4e^{i\left(\frac{7\pi}{6} - \frac{\pi}{2} - \alpha\right)} = 4e^{i\left(\frac{2\pi}{3} - \alpha\right)}$

Para $\frac{z_1}{z_3}$ ser um número imaginário puro:

$\frac{2\pi}{3} - \alpha = \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \Leftrightarrow -\alpha = \pi - \frac{2\pi}{3} + k\pi, k \in \mathbb{Z}$

$\Leftrightarrow -\alpha = -\frac{\pi}{6} + k\pi, k \in \mathbb{Z}$

$\Leftrightarrow \alpha = \frac{\pi}{6} + k\pi, k \in \mathbb{Z}$

10.

a) $z - \frac{i}{z} = 0 \Leftrightarrow z^2 - i = 0 \wedge z \neq 0 \Leftrightarrow z^2 = i \wedge z \neq 0$

As soluções desta equação são as raízes quadradas do número complexo $i = e^{i\frac{\pi}{2}}$, isto é, os valores de z_k que satisfazem $z_k = e^{i\left(\frac{\pi}{4} + k\pi\right)}, k \in \{0, 1\}$.

Se $k = 0$, então $z_0 = e^{i\frac{\pi}{4}}$.

Se $k = 1$, então $z_1 = e^{i\frac{5\pi}{4}}$.

C.S. $= \left\{e^{i\frac{\pi}{4}}, e^{i\frac{5\pi}{4}}\right\}$

b) $z^3 + (1 + \sqrt{3}i)z = 0 \Leftrightarrow z(z^2 + 1 + \sqrt{3}i) = 0$

$\Leftrightarrow z = 0 \vee z^2 = -1 - \sqrt{3}i$

Cálculo auxiliar

$w = -1 - \sqrt{3}i = 2e^{i\frac{4\pi}{3}}$

$\rho = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2$

$\text{tg}\,\theta = -\frac{\sqrt{3}}{-1} \wedge \theta \in 3.^{\circ}\text{Q} \Leftrightarrow \text{tg}\,\theta = \sqrt{3} \wedge \theta \in 3.^{\circ}\text{Q}$

Por exemplo, $\theta = \frac{4\pi}{3}$.

As soluções da equação $z^2 = -1 - \sqrt{3}i$ são as raízes quadradas do número complexo $2e^{i\frac{4\pi}{3}}$, isto é, os números complexos z_k tais que:

$z_k = \sqrt{2}e^{i\left(\frac{\frac{4\pi}{3} + 2k\pi}{2}\right)}, k \in \{0, 1\}$

Se $k = 0$, então $z_0 = \sqrt{2}e^{i\frac{2\pi}{3}}$.

Se $k = 1$, então $z_1 = \sqrt{2}e^{i\frac{5\pi}{3}}$.

C.S. $= \left\{0, \sqrt{2}e^{i\frac{2\pi}{3}}, \sqrt{2}e^{i\frac{5\pi}{3}}\right\}$

c) $z^3 - iz^2 - z + i = 0$

Cálculo auxiliar

	1	$-i$	-1	i
i		i	0	$-i$
	1	0	-1	$0 = r$

$\Leftrightarrow (z - i)(z^2 - 1) = 0$

$\Leftrightarrow z = i \vee z^2 = 1$

$\Leftrightarrow z = i \vee z = -1 \vee z = 1$

C.S. $= \{i, -1, 1\}$

d) $z^3 = \bar{z} \Leftrightarrow (\rho e^{i\theta})^3 = \rho e^{i(-\theta)}$

$\Leftrightarrow \rho^3 e^{i3\theta} = \rho e^{i(-\theta)}$

$\Leftrightarrow \begin{cases} \rho^3 = \rho \\ 3\theta = -\theta + 2k\pi, k \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} \rho(\rho^2 - 1) = 0 \\ 4\theta = 2k\pi, k \in \mathbb{Z} \end{cases}$

$\Leftrightarrow \begin{cases} \rho = 0 \\ \theta = \frac{k\pi}{2}, k \in \mathbb{Z} \end{cases} \vee \begin{cases} \rho = 1 \\ \theta = \frac{k\pi}{2}, k \in \mathbb{Z} \end{cases}$

$z = 0$ é uma das soluções

Se $k = 0$, então $z_0 = e^{i0} = 1$.

Se $k = 1$, então $z_1 = e^{i\frac{\pi}{2}} = i$.

Se $k = 2$, então $z_2 = e^{i\pi} = -1$.

Se $k = 3$, então $z_3 = e^{i\frac{3\pi}{2}} = -i$.

C.S. $= \{0, 1, i, -1, -i\}$

11.

a) $z_1 = 4\sqrt{3} + 4i$

Cálculo auxiliar

$\rho = \sqrt{(4\sqrt{3})^2 + 4^2} = \sqrt{64} = 8$

$\text{tg}\,\theta = \frac{4}{4\sqrt{3}} \wedge \theta \in 1.^{\circ}\text{Q} \Leftrightarrow \text{tg}\,\theta = \frac{\sqrt{3}}{3} \wedge \theta \in 1.^{\circ}\text{Q}$

Por exemplo, $\theta = \frac{\pi}{6}$.

As raízes de ordem 4 do número complexo $8e^{i\frac{\pi}{6}}$ são os valores z_k tais que:

$z_k = \sqrt[4]{8}\, e^{i\left(\frac{\frac{\pi}{6} + 2k\pi}{4}\right)}, k \in \{0, 1, 2, 3\}$

$\Leftrightarrow z_k = \sqrt[4]{8}\, e^{i\left(\frac{\pi}{24} + \frac{k\pi}{2}\right)}, k \in \{0, 1, 2, 3\}$

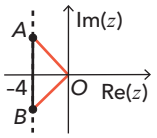
Se $k = 0$, então $z_0 = \sqrt[4]{8}\, e^{i\frac{\pi}{24}}$.

Se $k = 1$, então $z_1 = \sqrt[4]{8}\, e^{i\frac{13\pi}{24}}$.

Se $k = 2$, então $z_2 = \sqrt[4]{8}\, e^{i\frac{25\pi}{24}}$.

Se $k = 3$, então $z_3 = \sqrt[4]{8}\, e^{i\frac{37\pi}{24}}$.

b) **i.**



ii. Se $A_{\triangle[AOB]} = 12$, então $\frac{\overline{AB} \times 4}{2} = 12 \Leftrightarrow \overline{AB} = 6$.

Assim, $A(-4, 3)$ e $B(-4, -3)$. Logo, $z = -4 + 3i$.

12.

a) Se z_1 é solução da equação, então verifica $-i \times z_1 = -\bar{z}_1$.

$-i \times 16e^{i\frac{3\pi}{4}} = -16e^{i\left(-\frac{3\pi}{4}\right)}$

$\Leftrightarrow e^{i\left(-\frac{\pi}{2}\right)} \times 16e^{i\frac{3\pi}{4}} = 16e^{i\left(\pi - \frac{3\pi}{4}\right)}$

$\Leftrightarrow 16e^{i\left(-\frac{\pi}{2} + \frac{3\pi}{4}\right)} = 16e^{i\frac{\pi}{4}}$

$\Leftrightarrow 16e^{i\frac{\pi}{4}} = 16e^{i\frac{\pi}{4}}$, que é uma proposição verdadeira.

b) As raízes de ordem 4 de $z_1 = 16e^{i\frac{3\pi}{4}}$ são os valores z_k tais que:

$z_k = \sqrt[4]{16}e^{i\left(\frac{\frac{3\pi}{4} + 2k\pi}{4}\right)}, k \in \{0, 1, 2, 3\}$

$\Leftrightarrow z_k = 2e^{i\left(\frac{3\pi}{16} + \frac{k\pi}{2}\right)}, k \in \{0, 1, 2, 3\}$

Os afixos destas quatro raízes de ordem 4 (A, B, C e D) são vértices de um quadrado de diagonal 4. Assim, sendo ℓ o lado do quadrado, vem que:

$\ell^2 + \ell^2 = 4^2 \Leftrightarrow 2\ell^2 = 16 \Leftrightarrow \ell^2 = 8$

Logo, $A_{[ABCD]} = 8$ u.a.

13.

a) $\frac{(1 + 3i)z_1 - i^{49} + i^{100}}{1 + 4i} = \frac{(1 + 3i)(2 + i) - i^1 + i^0}{1 + 4i} =$

$= \frac{2 + i + 6i + 3i^2 - i + 1}{1 + 4i} =$

$= \frac{6i}{1 + 4i} = \frac{6i(1 - 4i)}{1 + 16} =$

$= \frac{6i - 24i^2}{17} = \frac{24}{17} + \frac{6}{17}i$

b) As raízes de ordem 4 de um número complexo formam um quadrado inscrito numa circunferência centrada na origem do referencial.

Se z_1 é uma das raízes de ordem 4 de um número complexo z, então as restantes raízes de ordem 4 do mesmo número complexo são:

$iz_1 = i(2 + i) = 2i + i^2 = -1 + 2i$

$-z_1 = -(2 + i) = -2 - i$

$-iz_1 = i(-2 - i) = -2i - i^2 = 1 - 2i$

c) $iz_2 = e^{i\frac{\pi}{2}} \times e^{i\frac{n\pi}{10}} = e^{i\left(\frac{\pi}{2}+\frac{n\pi}{10}\right)}$

Para que iz_2 seja um número real:

$\frac{\pi}{2} + \frac{n\pi}{10} = k\pi,\ k \in \mathbb{Z} \Leftrightarrow \frac{n\pi}{10} = -\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

$\Leftrightarrow n = -5 + 10k,\ k \in \mathbb{Z}$

Se $k = 0$, então $n = -5 \notin \mathbb{N}$.

Se $k = 1$, então $n = 5 \in \mathbb{N}$.

Logo, $n = 5$.

14. Opção (A)

Seja $z = x + yi$

$z - \bar{z} = 0 \Leftrightarrow (x + yi) - (x - yi) = 0$

$\Leftrightarrow 2yi = 0$

$\Leftrightarrow y = 0$

Condição que define o eixo real:

$\text{Re}(z) = 0 \Leftrightarrow \text{Re}(x + yi) = 0$

$\Leftrightarrow x = 0$

$|z| = 0 \Leftrightarrow |x + yi| = 0$

$\Leftrightarrow \sqrt{x^2 + y^2} = 0$

$\Leftrightarrow x^2 + y^2 = 0$

$z + \bar{z} = 0 \Leftrightarrow (x + yi) + (x - yi) = 0$

$\Leftrightarrow 2x = 0$

$\Leftrightarrow x = 0$

15.

a) O círculo representado tem centro O e raio 2, visto que $|z| = 2$. Sendo o hexágono regular e centrado na origem, então os seis vértices são os afixos das raízes índice 6 de um certo número complexo e os seus argumentos diferem de $\frac{2\pi}{6} = \frac{\pi}{3}$.

Assim, uma condição que defina o conjunto de pontos a sombreado é:

$|z| \leq 2 \wedge \frac{\pi}{5} \leq \arg(z) \leq \frac{13\pi}{15}$

Cálculo auxiliar

$\frac{\pi}{5} + 2 \times \frac{\pi}{3} = \frac{13\pi}{15}$

b) Cálculo do raio da circunferência:

$r^2 = 2^2 + 2^2 \Leftrightarrow r^2 = 8$

$\Leftrightarrow r^2 = \sqrt{8}, r > 0$

$\Leftrightarrow r = 2\sqrt{2}$

$|z| > 2\sqrt{2} \wedge |z - 2| = |z - 2i|$

c) Sendo P_1 o ponto de coordenadas (0, 2), o número complexo do qual P_1 é afixo é $z_1 = 2i = 2e^{i\frac{\pi}{2}}$.

As restantes raízes cúbicas são:

$z_2 = 2e^{i\left(\frac{\pi}{2}+\frac{2\pi}{3}\right)} = 2e^{i\frac{7\pi}{6}} = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - i$

$z_3 = 2e^{i\left(\frac{7\pi}{6}+\frac{2\pi}{3}\right)} = 2e^{i\frac{11\pi}{6}} = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i$

$\text{Im}(z_1) = \text{Im}(z_2) = -1$

Assim, uma condição que define o domínio plano é $|z| \leq 2 \wedge \text{Im}(z) \leq -1$.

d) O lado do quadrado é $\sqrt{2}$. Logo, o raio é $\frac{\sqrt{2}}{2}$.

$|z| \geq \frac{\sqrt{2}}{2} \wedge -\frac{\pi}{4} \leq \arg(z + 1) \leq \frac{\pi}{4} \wedge$

$\wedge \frac{3\pi}{4} \leq \arg(z - 1) \leq \frac{5\pi}{4}$

16.

a) $\bar{z} = -z$

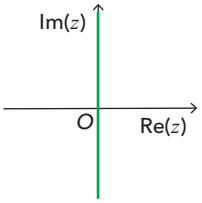
Seja $z = x + yi$. Então:

$x - yi = -(x + yi) \Leftrightarrow x - yi = -x - yi$

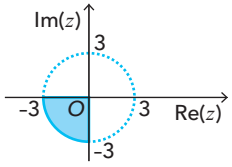
$\Leftrightarrow 2x = 0$

$\Leftrightarrow x = 0$

Condição que define todo o eixo imaginário.

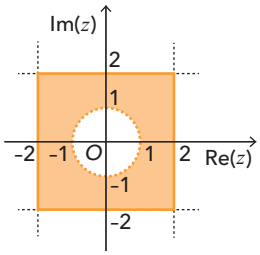


b) $|z| \leq 3 \wedge \text{Im}(z) \leq 0 \wedge \text{Re}(z) \leq 0$

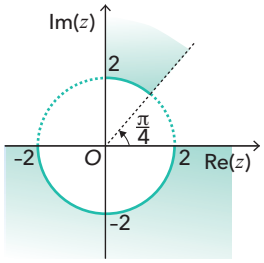


c) $|\text{Re}(z)| \leq 2 \wedge |\text{Im}(z)| \leq 2 \wedge |z| > 1$

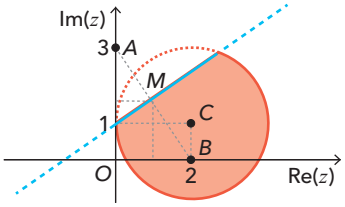
$\Leftrightarrow -2 \leq \text{Re}(z) \leq 2 \wedge -2 \leq \text{Im}(z) \leq 2 \wedge |z| > 1$



d) $\left[\frac{\pi}{4} \leq \arg(z) \leq \frac{\pi}{2} \vee \text{Im}(z) \leq 0\right] \wedge |z| \geq 2$



e) $|z - 3i| \geq |z - 2| \wedge |z - 2 - i| \leq 2$



17.

a) $w = \frac{-1+i}{i^{21}} = \frac{-1+i}{i} =$

$= \frac{(-1+i)\times(-i)}{i\times(-i)} =$

$= \frac{i-i^2}{-i^2} =$

$= \frac{1+i}{1} =$

$= 1+i =$

$= \sqrt{2}e^{i\frac{\pi}{4}}$

$w \neq z_1$, pois $\arg(w) = \frac{\pi}{4} + 2k\pi,\ k \in \mathbb{Z}$ e um argumento de z_1 é $\frac{\pi}{5}$.

$w \neq z_2$, pois $|w| = \sqrt{2}$ e $|z_2| = 2$.

b) Seja $z_2 = 2e^{i\theta}$ e $z_1 = \rho e^{i\frac{\pi}{5}}$

$z_1 \times z_2 = \rho e^{i\frac{\pi}{5}} \times 2e^{i\theta} = 2\rho e^{i\left(\theta+\frac{\pi}{5}\right)}$

Para que $z_1 \times z_2$ seja um número imaginário puro, tem que acontecer:

$\arg(z_1 \times z_2) = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

Assim:

$\theta + \frac{\pi}{5} = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

$\Leftrightarrow \theta = \frac{\pi}{2} - \frac{\pi}{5} + k\pi,\ k \in \mathbb{Z}$

$\Leftrightarrow \theta = \frac{3\pi}{10} + k\pi,\ k \in \mathbb{Z}$

18. $\frac{1+\sqrt{2}+i}{1+\sqrt{2}-i} = \frac{(1+\sqrt{2}+i)(1+\sqrt{2}+i)}{(1+\sqrt{2}-i)(1+\sqrt{2}+i)} =$

$= \frac{(1+\sqrt{2}+i)^2}{(1+\sqrt{2})^2 - i^2} =$

$= \frac{(1+\sqrt{2})^2 + 2(1+\sqrt{2})i + i^2}{(1+\sqrt{2})^2 + 1} =$

$= \frac{1+2\sqrt{2}+2-1+(2+2\sqrt{2})i}{1+2\sqrt{2}+2+1} =$

$= \frac{(2\sqrt{2}+2)(1+i)}{2\sqrt{2}+4} =$

$= \frac{(2\sqrt{2}+2)(2\sqrt{2}-4)}{(2\sqrt{2}+4)(2\sqrt{2}-4)}(1+i) =$

$= \frac{4\sqrt{4}-8\sqrt{2}+4\sqrt{2}-8}{(2\sqrt{2})^2-4^2}(1+i) =$

$= \frac{-4\sqrt{2}}{4\times 2-16}(1+i) =$

$= \frac{-4\sqrt{2}}{-8}(1+i) =$

$= \frac{\sqrt{2}}{2}(1+i) =$

$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i =$

$= \cos\left(\frac{\pi}{4}\right) + i\,\mathrm{sen}\left(\frac{\pi}{4}\right) =$

$= e^{i\frac{\pi}{4}}$

19. **1º processo**

$\frac{1+\cos\theta+i\,\mathrm{sen}\,\theta}{1+\cos\theta-i\,\mathrm{sen}\,\theta} = \frac{1+e^{i\theta}}{1+\overline{e^{i\theta}}} = \frac{1+e^{i\theta}}{1+e^{i(-\theta)}} =$

$= \frac{1+e^{i\theta}}{1+\frac{1}{e^{i\theta}}} =$

$= \frac{1+e^{i\theta}}{\frac{e^{i\theta}+1}{e^{i\theta}}} =$

$= \frac{e^{i\theta}(1+e^{i\theta})}{e^{i\theta}+1} =$

$= e^{i\theta}$

2º processo

$\frac{1+\cos\theta+i\,\mathrm{sen}\,\theta}{1+\cos\theta-i\,\mathrm{sen}\,\theta} =$

$= \frac{(1+\cos\theta+i\,\mathrm{sen}\,\theta)(1+\cos\theta+i\,\mathrm{sen}\,\theta)}{(1+\cos\theta-i\,\mathrm{sen}\,\theta)(1+\cos\theta+i\,\mathrm{sen}\,\theta)} =$

$= \frac{(1+\cos\theta+i\,\mathrm{sen}\,\theta)^2}{(1+\cos\theta)^2\,(i\,\mathrm{sen}\,\theta)^2} =$

$= \frac{(1+\cos\theta)^2 + 2(1+\cos\theta)i\,\mathrm{sen}\,\theta + (i\,\mathrm{sen}\,\theta)^2}{(1+\cos\theta)^2 + \mathrm{sen}^2\theta} =$

$= \frac{(1+\cos\theta)^2 + i^2\,\mathrm{sen}^2\theta + 2(1+\cos\theta)\,\mathrm{sen}\,\theta\, i}{1+2\cos\theta+\underbrace{\cos^2\theta+\mathrm{sen}^2\theta}_{1}} =$

$= \frac{(1+\cos\theta)^2 - \mathrm{sen}^2\theta + 2(1+\cos\theta)\,\mathrm{sen}\,\theta\, i}{2+2\cos\theta} =$

$= \frac{1+2\cos\theta+\cos^2\theta-\mathrm{sen}^2\theta}{2(1+\cos\theta)} + \frac{2(1+\cos\theta)\,\mathrm{sen}\,\theta\, i}{2(1+\cos\theta)} =$

$= \frac{\cos^2\theta+\cos^2\theta+2\cos\theta}{2(1+\cos\theta)} + i\,\mathrm{sen}\,\theta =$

$= \frac{2\cos^2\theta+2\cos\theta}{2(1+\cos\theta)} + i\,\mathrm{sen}\,\theta =$

$= \frac{2\cos\theta+(\cos\theta+1)}{2(1+\cos\theta)} + i\,\mathrm{sen}\,\theta =$

$= \cos\theta + i\,\mathrm{sen}\,\theta =$

$= e^{i\theta}$

20.

a) $z^3 - z^2 + 9z - 9 = 0 \Leftrightarrow (z-1)(z^2+9) = 0$

$\Leftrightarrow z-1 = 0 \vee z^2+9 = 0$

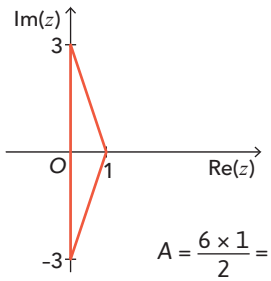
$\Leftrightarrow z = 1 \vee z^2 = -9$

$\Leftrightarrow z = 1 \vee z = 3i \vee z = -3i$

$\Leftrightarrow z = e^{i0} \vee z = 3e^{i\frac{\pi}{2}} \vee z = 3e^{i\frac{3\pi}{2}}$

Cálculo auxiliar

	1	−1	9	−9
1		1	0	9
	1	0	9	0



$A = \frac{6\times 1}{2} = 3$ u.a.

b) $|z_1 + z_2|^2 = \left|1 + e^{i\frac{\pi}{5}}\right|^2 =$

$= \left|1 + \cos\left(\frac{\pi}{5}\right) + i\,\text{sen}\left(\frac{\pi}{5}\right)\right|^2 =$

$= \left(1 + \cos\left(\frac{\pi}{5}\right)\right)^2 + \left(\text{sen}\left(\frac{\pi}{5}\right)\right)^2 =$

$= 1 + 2\cos\left(\frac{\pi}{5}\right) + \cos^2\left(\frac{\pi}{5}\right) + \text{sen}^2\left(\frac{\pi}{5}\right) =$

$= 2 + 2\cos\left(\frac{\pi}{5}\right)$

21. Opção (B)

Como w tem o seu afixo na parte negativa do eixo imaginário, então w é do tipo $w = \rho e^{i\left(-\frac{\pi}{2}\right)}$. Logo, as raízes cúbicas de w são os valores w_k tais que:

$w_k = \sqrt[3]{\rho} e^{i\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right)}$, $k \in \{0, 1, 2\}$

Se $k = 0$, então $w_0 = \sqrt[3]{\rho} e^{i\left(-\frac{\pi}{6}\right)}$.

Se $k = 1$, então $w_1 = \sqrt[3]{\rho} e^{i\frac{\pi}{2}}$.

Se $k = 2$, então $w_2 = \sqrt[3]{\rho} e^{i\frac{7\pi}{6}}$.

Observe-se que o afixo de w_1 pertence à parte positiva do eixo imaginário. Logo, das opções apresentadas, apenas a opção (B) satisfaz esta condição.

22.

a) $i^{11} \times z^3 = (z_2 \times \bar{z}_1)^2 \Leftrightarrow i^3 \times z^3 = \left(2e^{i\frac{\pi}{6}} \times 2e^{i\left(-\frac{\pi}{3}\right)}\right)^2$

Cálculo auxiliar

$\rho_1 = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

$\text{tg}\,\theta = \sqrt{3} \wedge \theta \in 1.^{\circ}\text{Q}$

Por exemplo, $\theta = \frac{\pi}{3}$.

$\Leftrightarrow -i \times z^3 = \left(4e^{i\left(\frac{\pi}{6} - \frac{\pi}{3}\right)}\right)^2$

$\Leftrightarrow z^3 = \frac{\left(4e^{i\left(-\frac{\pi}{6}\right)}\right)^2}{-i}$

$\Leftrightarrow z^3 = \frac{16e^{i\left(-\frac{\pi}{3}\right)}}{e^{i\left(-\frac{\pi}{2}\right)}}$

$\Leftrightarrow z^3 = 16e^{i\left(-\frac{\pi}{3} + \frac{\pi}{2}\right)}$

$\Leftrightarrow z^3 = 16e^{i\frac{\pi}{6}}$

As soluções da equação são as raízes cúbicas do número complexo $16e^{i\frac{\pi}{6}}$, isto é, os números complexos z_k tais que:

$z_k = \sqrt[3]{16} e^{i\left(\frac{\frac{\pi}{6} + 2k\pi}{3}\right)}$, $k \in \{0, 1, 2\}$

$\Leftrightarrow z_k = \sqrt[3]{16} e^{i\left(\frac{\pi}{18} + \frac{2k\pi}{3}\right)}$, $k \in \{0, 1, 2\}$

Se $k = 0$, então $z_0 = \sqrt[3]{16} e^{i\frac{\pi}{18}}$.

Se $k = 1$, então $z_1 = \sqrt[3]{16} e^{i\frac{13\pi}{18}}$.

Se $k = 2$, então $z_2 = \sqrt[3]{16} e^{i\frac{25\pi}{18}}$.

C.S. $= \left\{\sqrt[3]{16} e^{i\frac{\pi}{18}}, \sqrt[3]{16} e^{i\frac{13\pi}{18}}, \sqrt[3]{16} e^{i\frac{25\pi}{18}}\right\}$

b) $\left(\frac{z_1}{z_2}\right)^{2n} = \left(\frac{2e^{i\frac{\pi}{3}}}{2e^{i\frac{\pi}{6}}}\right)^{2n} = \left(e^{i\left(\frac{\pi}{3} - \frac{\pi}{6}\right)}\right)^{2n} = e^{i\left(2n \times \frac{\pi}{6}\right)} = e^{i\left(\frac{n\pi}{3}\right)}$

Para que $\left(\frac{z_1}{z_2}\right)^{2n}$ seja um número real positivo, então $\arg\left(\left(\frac{z_1}{z_2}\right)^{2n}\right) = 2k\pi$, $k \in \mathbb{Z}$, ou seja, $\frac{\pi}{3} n = 2k\pi$, $k \in \mathbb{Z}$

$\Leftrightarrow n = 6k$, $k \in \mathbb{Z}$.

Para $k = 1$, $n = 6$ é o menor valor natural positivo de n que verifica o pretendido.

c) $w = a + bi + (1 - 4i + (2i)^2) =$

$= a + bi + 1 - 4i - 4 =$

$= (a - 3) + (b - 4)i$

Como o afixo de w pertence à bissetriz dos quadrantes ímpares $\text{Re}(w) = \text{Im}(w)$, ou seja, $a - 3 = b - 4$.

Como $|w| = 5\sqrt{2}$, tem-se que:

$\sqrt{(a-3)^2 + (b-4)^2} = 5\sqrt{2}$

$\begin{cases} a - 3 = b - 4 \\ (a-3)^2 + (b-4)^2 = 25 \times 2 \end{cases}$

$\Leftrightarrow \begin{cases} a = b - 1 \\ (b-4)^2 + (b-4)^2 = 50 \end{cases}$

$\Leftrightarrow \begin{cases} a = b - 1 \\ 2(b-4)^2 = 50 \end{cases} \Leftrightarrow \begin{cases} \text{———} \\ (b-4)^2 = 25 \end{cases}$

$\Leftrightarrow \begin{cases} a = b - 1 \\ b - 4 = 5 \end{cases} \vee \begin{cases} a = b - 1 \\ b - 4 = -5 \end{cases}$

$\Leftrightarrow \begin{cases} a = 9 - 1 \\ b = 9 \end{cases} \vee \begin{cases} a = -1 - 1 \\ b = -1 \end{cases}$

$\Leftrightarrow \begin{cases} a = 8 \\ b = 9 \end{cases} \vee \begin{cases} a = -2 \\ b = -1 \end{cases}$

Assim, $z_3 = 8 + 9i \vee z_3 = -2 - i$.

23.

a) $\frac{w^4 + 2}{i^{15}} = \frac{\left(\sqrt{2}e^{i\frac{\pi}{3}}\right)^4 + 2}{i^3} = \frac{(\sqrt{2})^4 e^{i\frac{4\pi}{3}} + 2}{-i} =$

$= \frac{4\left(\cos\left(\frac{4\pi}{3}\right) + i\,\text{sen}\left(\frac{4\pi}{3}\right)\right) + 2}{-i} =$

$= \frac{4\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)}{-i} =$

$= \frac{2 - 2\sqrt{3}i + 2}{-i} =$

$= \frac{-2\sqrt{3}i \times i}{-i \times i} =$

$= \frac{-2\sqrt{3}i^2}{-i^2} =$

$= \frac{2\sqrt{3}}{1} =$

$= 2\sqrt{3}$

$2\sqrt{3}$ é um número real, como queríamos demonstrar.

b) Seja $z = -1 + i$.

Cálculo auxiliar

$\rho = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\text{tg}\,\theta = -\frac{1}{-1} \wedge \theta \in 2.^{\circ}\,Q \Leftrightarrow \text{tg}\,\theta = -1 \wedge \theta \in 2.^{\circ}\,Q$

Por exemplo, $\theta = \frac{3\pi}{4}$.

Como $\rho = \sqrt{2}$, $\sqrt{2}$ é o raio da circunferência representada.

As raízes cúbicas de $w = \sqrt{2}e^{i\frac{\pi}{3}}$ são os valores w_k tais que:

$w_k = \sqrt[3]{\sqrt{2}}e^{i\left(\frac{\frac{\pi}{3} + 2k\pi}{3}\right)}$, $k \in \{0, 1, 2\}$

$\Leftrightarrow w_k = \sqrt[6]{2}e^{i\left(\frac{\pi}{9} + \frac{2k\pi}{3}\right)}$, $k \in \{0, 1, 2\}$

Se $k = 0$, então $w_0 = \sqrt[6]{2}e^{i\frac{\pi}{9}}$.

Assim, $\arg(w_0) = \frac{\pi}{9}$.

Logo, uma condição em $\mathbb{C}$ que defina a região assinalada é:

$|z| \leq \sqrt{2} \wedge \frac{\pi}{9} \leq \arg(z) \leq \frac{3\pi}{4}$

24.

a) $\left|\frac{z}{i}\right| < 3 \wedge \arg(z) = \frac{k\pi}{2}, k \in \mathbb{Z}$

$\Leftrightarrow \frac{|z|}{|i|} < 3 \wedge \arg(z) = \frac{k\pi}{2}, k \in \mathbb{Z}$

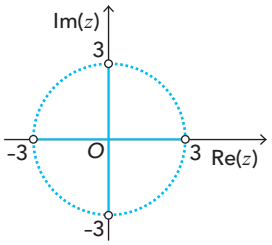
$\Leftrightarrow |z| < 3 \wedge \arg(z) = \frac{k\pi}{2}, k \in \mathbb{Z}$

Se $k = 0$, $\arg(z) = 0$.

Se $k = 1$, $\arg(z) = \frac{\pi}{2}$.

Se $k = 2$, $\arg(z) = \pi$.

Se $k = 3$, $\arg(z) = \frac{3\pi}{2}$.



b) $\text{Im}(z - \bar{z}) > z \times \bar{z} \wedge |z - \bar{z}| = 4$

Seja $z = x + yi$:

$\text{Im}((x + yi) - (x - yi)) > (x + yi) \times (x - yi) \wedge$

$\wedge |(x + yi) - (x - yi)| = 4$

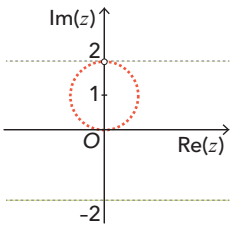
$\Leftrightarrow \text{Im}(2yi) > x^2 - (yi)^2 \wedge |2yi| = 4$

$\Leftrightarrow 2y > x^2 + y^2 \wedge (2y = 4 \vee 2y = -4)$

$\Leftrightarrow x^2 + y^2 - 2y < 0 \wedge (y = 2 \vee y = -2)$

$\Leftrightarrow x^2 + y^2 - 2y + 1 < 1 \wedge (y = 2 \vee y = -2)$

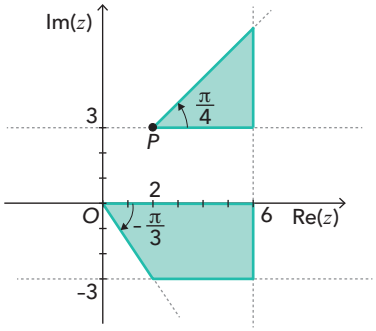
$\Leftrightarrow x^2 + (y - 1)^2 < 1 \wedge (y = 2 \vee y = -2)$



Não existe nenhum ponto do plano complexo que satisfaça a condição pretendida.

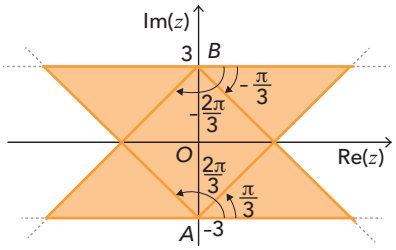
c) $\left[0 \leq \arg(z - 2 - 3i) \leq \frac{\pi}{4} \wedge \text{Re}(z) \leq 6\right] \vee$

$\vee \left[-\frac{\pi}{3} \leq \arg(z) \leq 0 \wedge \text{Im}(z) \geq -3 \wedge \text{Re}(z) \leq 6\right]$



d) $\left[\frac{\pi}{3} \leq \arg(z + 3i) \leq \frac{2\pi}{3} \wedge |z| \leq |z - 6i|\right] \vee$

$\vee \left[-\frac{2\pi}{3} \leq \arg(z - 3i) \leq -\frac{\pi}{3} \wedge |z| \leq |z + 6i|\right]$



Exercícios–Tipo Exame – páginas 86 a 88

1. $z^3 - 4z \times \bar{z} + 10 - 2i = 0$

$\Leftrightarrow (\sqrt{2}e^{i\theta})^3 - 4 \times 2 + 10 - 2i = 0$

$\Leftrightarrow (\sqrt{2})^3 e^{i(3\theta)} = -2 + 2i$

$\Leftrightarrow \sqrt{8}e^{i(3\theta)} = \sqrt{8}e^{i\frac{3\pi}{4}}$

$\Leftrightarrow \underbrace{\sqrt{8} = \sqrt{8}}_{\text{condição universal}} \wedge 3\theta = \frac{3\pi}{4} + 2k\pi, k \in \mathbb{Z}$

$\Leftrightarrow \theta = \frac{\pi}{4} + \frac{2k\pi}{3}, k \in \mathbb{Z}$

Como $\theta \in \left]0, \frac{\pi}{2}\right[$, $\theta = \frac{\pi}{4}$.

Assim, $z = \sqrt{2}e^{i\frac{\pi}{4}}$.

Como z é uma das raízes sextas de um certo número complexo w, então $z^6 = w$.

$w = z^6 = \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^6 =$

$= (\sqrt{2})^6 e^{i\frac{6\pi}{4}} =$

$= 8e^{i\frac{3\pi}{2}} =$

$= -8i$

Cálculos auxiliares

- $z \times \bar{z} = (\sqrt{2}e^{i\theta}) \times (\sqrt{2}e^{-i\theta}) =$
  $= (\sqrt{2})^2 e^{i(\theta - \theta)} =$
  $= 2e^{i0} =$
  $= 2$
- $-2 + 2i = \sqrt{8}e^{i\frac{3\pi}{4}}$
  $|-2 + 2i| = \sqrt{(-2)^2 + 2^2} = \sqrt{8}$
  $\text{tg}\,\theta = \frac{2}{-2} \wedge \alpha \in 2.^{\circ}\,Q$
  $\theta = \frac{3\pi}{4}$, por exemplo.

2. Opção (D)

$$w = z \times e^{i\frac{4\pi}{6}} = (2+i) \times e^{i\frac{2\pi}{3}} =$$
$$= (2+i)\left(\cos\left(\frac{2\pi}{3}\right) + i\,\text{sen}\left(\frac{2\pi}{3}\right)\right) =$$
$$= (2+i)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) =$$
$$= -1 + \sqrt{3}\,i - \frac{1}{2}i - \frac{\sqrt{3}}{2} =$$
$$= \left(-1 - \frac{\sqrt{3}}{2}\right) + \left(\sqrt{3} - \frac{1}{2}\right)i$$

$\text{Re}(w) = -1 - \frac{\sqrt{3}}{2}$ e $\text{Im}(w) = \sqrt{3} - \frac{1}{2}$

$\frac{w + \overline{w}}{2} = \text{Re}(w) = -1 - \frac{\sqrt{3}}{2}$

$\frac{w - \overline{w}}{2i} = \text{Im}(w) = \sqrt{3} - \frac{1}{2}$

3. $w = \left(\frac{\sqrt{6}\left(4e^{i\frac{\pi}{3}} + 2e^{i\frac{5\pi}{3}}\right)}{2(2+i)^2 - 2i^{-3}}\right)^{12} = \left(\frac{\sqrt{6} \times \sqrt{12}e^{i\frac{\pi}{6}}}{\sqrt{72}e^{i\frac{\pi}{4}}}\right)^{12} =$

$$= \left(e^{i\left(\frac{\pi}{6} - \frac{\pi}{4}\right)}\right)^{12} =$$
$$= \left(e^{i\left(-\frac{2\pi}{24}\right)}\right)^{12} =$$
$$= e^{i\left(-\frac{\pi}{12} \times 12\right)} =$$
$$= e^{i(-\pi)} =$$
$$= e^{i\pi}$$

Cálculos auxiliares

- $4e^{i\frac{\pi}{3}} + 2e^{i\frac{5\pi}{3}} = 4\left(\cos\left(\frac{\pi}{3}\right) + i\,\text{sen}\left(\frac{\pi}{3}\right)\right) + 2\left(\cos\left(\frac{5\pi}{3}\right) + i\,\text{sen}\left(\frac{5\pi}{3}\right)\right) =$
  $= 4\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) =$
  $= 2 + 2\sqrt{3}\,i + 1 - \sqrt{3}\,i =$
  $= 3 + \sqrt{3}\,i =$
  $= \sqrt{12}\,e^{i\frac{\pi}{6}}$
- $|3 + \sqrt{3}\,i| = \sqrt{9+3} = \sqrt{12}$
  $\text{tg}\,\theta = \frac{\sqrt{3}}{3} \wedge \theta \in 1.^{\circ}\,Q$
  $\theta = \frac{\pi}{6}$, por exemplo.
- $2(2+i)^2 - 2i^{-3} = 2(4 + 4i + (-1)) - 2i^{-3\times 4} =$
  $= 2(3 + 4i) - 2i =$
  $= 6 + 8i - 2i =$
  $= 6 + 6i =$
  $= \sqrt{72}e^{i\frac{\pi}{4}}$

$z^4 = w \Leftrightarrow z^4 = e^{i\pi}$

Pretendemos determinar as raízes quartas de $e^{i\pi}$, que são do tipo:

$$z_k = \sqrt[4]{1} \times e^{i\frac{\pi + 2k\pi}{4}},\ k \in \{0, 1, 2, 3\}$$

$k = 0 \leadsto z_0 = 1 \times e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

$k = 1 \leadsto z_1 = 1 \times e^{i\frac{3\pi}{4}} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

$k = 2 \leadsto z_2 = 1 \times e^{i\frac{5\pi}{4}} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

$k = 3 \leadsto z_3 = 1 \times e^{i\frac{7\pi}{4}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

$\text{C.S.} = \left\{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right\}$

4.

4.1 Opção (A)

$\rho = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

Logo, 5 é o raio da circunferência de centro na origem e que passa em *A* e em *B*.

Como o comprimento do arco *AB* é $\frac{5\pi}{6}$, tem-se:

$\frac{5\pi}{6} = \alpha \times 5 \Leftrightarrow \alpha = \frac{\pi}{6}$

Então, $\frac{2\pi}{n} = \frac{\pi}{6}$, logo, $n = 12$.

Seja z_B o número complexo cujo afixo é *B*.

$$z_B = z \times e^{i\frac{\pi}{6}} = (3+4i)\left(\cos\left(\frac{\pi}{6}\right) + i\,\text{sen}\left(\frac{\pi}{6}\right)\right) =$$
$$= (3+4i)\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) =$$
$$= \frac{3\sqrt{3}}{2} + \frac{3}{2}i + 2\sqrt{3}i + 2i^2 =$$
$$= -2 + \frac{3}{2}\sqrt{3} + i\left(\frac{3}{2} + 2\sqrt{3}\right) =$$
$$= \frac{3\sqrt{3} - 4}{2} + i\,\frac{3 + 4\sqrt{3}}{2}$$

Assim, as coordenadas de *B* são $\left(\frac{3\sqrt{3} - 4}{2}, \frac{3 + 4\sqrt{3}}{2}\right)$.

4.2 $\text{Re}(z \times w) = \frac{1}{4} \Leftrightarrow \text{Re}\left(6e^{i\alpha} \times \frac{1}{3}e^{i\alpha}\right) = \frac{1}{4}$

$$\Leftrightarrow \text{Re}\left(6 \times \frac{1}{3}e^{i(\alpha+\alpha)}\right) = \frac{1}{4}$$
$$\Leftrightarrow \text{Re}(2e^{i(2\alpha)}) = \frac{1}{4}$$
$$\Leftrightarrow \text{Re}(2\cos(2\alpha) + 2\,\text{sen}(2\alpha)i) = \frac{1}{4}$$
$$\Leftrightarrow 2\cos(2\alpha) = \frac{1}{4}$$
$$\Leftrightarrow \cos(2\alpha) = \frac{1}{8}$$
$$\Leftrightarrow \cos^2\alpha - \text{sen}^2\alpha = \frac{1}{8}$$
$$\Leftrightarrow \cos^2\alpha - (1 - \cos^2\alpha) = \frac{1}{8}$$
$$\Leftrightarrow 2\cos^2\alpha - 1 = \frac{1}{8}$$
$$\Leftrightarrow 2\cos^2\alpha = \frac{9}{8}$$
$$\Leftrightarrow \cos^2\alpha = \frac{9}{16}$$
$$\Leftrightarrow \cos\alpha = \pm\frac{3}{4}$$

Como $\alpha \in \left]\pi, \frac{3\pi}{2}\right[$, $\cos\alpha = -\frac{3}{4}$.

Assim:

$\text{sen}^2\alpha = 1 - \cos^2\alpha = 1 - \frac{9}{16} = \frac{7}{16}$

Então, $\text{sen}\,\alpha = \pm\frac{\sqrt{7}}{4}$.

Como $\alpha \in \left]\pi, \frac{3\pi}{2}\right[$, $\text{sen}\,\alpha = -\frac{\sqrt{7}}{4}$.

$z = 6e^{i\alpha} \Leftrightarrow z = 6(\cos\alpha + i\,\text{sen}\,\alpha)$

$$\Leftrightarrow z = 6 \times \left(-\frac{3}{4}\right) + 6 \times \left(-\frac{\sqrt{7}}{4}\right)i$$
$$\Leftrightarrow z = -\frac{9}{2} - \frac{3\sqrt{7}}{2}i$$

5. Opção (C)

Como o triângulo [*OAB*] é equilátero, então os pontos *A* e *B* são os afixos de dois números complexos com o mesmo módulo e cujo argumento difere de $\frac{\pi}{3}$ rad.

Assim, sendo *A* o afixo de um número complexo $z = \rho e^{i\theta}$, então *B* é o afixo do número complexo $z' = \rho e^{i\left(\theta + \frac{\pi}{3}\right)}$.

Tem-se que:

$$\rho e^{i\left(\theta + \frac{\pi}{3}\right)} = \rho e^{i\theta} \times e^{i\frac{\pi}{3}} =$$
$$= \rho e^{i\theta} \times \left(\cos\left(\frac{\pi}{3}\right) + i\,\text{sen}\left(\frac{\pi}{3}\right)\right) =$$
$$= z \times \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

6.
$$z_1 = \sqrt{2}e^{i\frac{\pi}{4}} + i^{2031} =$$
$$= \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\,\text{sen}\left(\frac{\pi}{4}\right)\right) + i^{4\times 507+3} =$$
$$= \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) + i^3 =$$
$$= 1 + i - i =$$
$$= 1$$
$$\overline{z}_2 = e^{i(-\theta)}$$

Assim:

$$z_1 + \overline{z}_2 = 1 + e^{i(-\theta)} =$$
$$= 1 + \cos(-\theta) + i\,\text{sen}(-\theta) =$$
$$= 1 + \cos\theta - i\,\text{sen}\,\theta$$
$$|z_1 + \overline{z}_2|^2 = \left(\sqrt{(1+\cos\theta)^2 + (-\text{sen}\,\theta)^2}\right)^2 =$$
$$= 1 + 2\cos\theta + \underbrace{\cos^2\theta + \text{sen}^2\theta}_{1} =$$
$$= 1 + 2\cos\theta + 1 =$$
$$= 2 + 2\cos\theta$$

Como, para qualquer valor real de θ, se tem:

$$-1 \leq \cos\theta \leq 1$$

então:

$$-2 \leq 2\cos\theta \leq 2$$

e:

$$0 \leq 2 + 2\cos\theta \leq 4$$

isto é:

$$|z_1 + \overline{z}_2|^2 \in [0, 4]$$

7. Seja $z = \rho e^{i\theta}$.

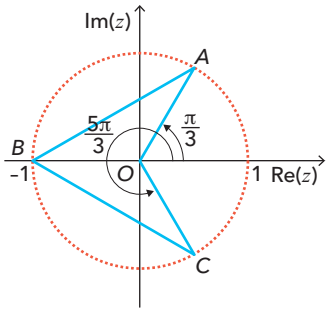
$$z^2 = -\overline{z} \Leftrightarrow (\rho e^{i\theta})^2 = -(\overline{\rho e^{i\theta}})$$
$$\Leftrightarrow \rho^2 e^{i(2\theta)} = -(\rho e^{i(-\theta)})$$
$$\Leftrightarrow \rho^2 e^{i(2\theta)} = \rho e^{i(-\theta+\pi)}$$
$$\Leftrightarrow \rho^2 = \rho \wedge 2\theta = -\theta + \pi + 2k\pi,\ k \in \mathbb{Z}$$
$$\Leftrightarrow \rho^2 - \rho = 0 \wedge 3\theta = \pi + 2k\pi,\ k \in \mathbb{Z}$$
$$\Leftrightarrow \rho(\rho - 1) = 0 \wedge \theta = \frac{\pi}{3} + \frac{2k\pi}{3},\ k \in \mathbb{Z}$$
$$\Leftrightarrow \rho = 0 \vee \left(\rho = 1 \wedge \theta = \frac{\pi}{3} + \frac{2k\pi}{3},\ k \in \mathbb{Z}\right)$$
$$\Leftrightarrow z = 0 \vee \left(\rho = 1 \wedge \theta = \frac{\pi}{3} + \frac{2k\pi}{3},\ k \in \mathbb{Z}\right)$$

$k = 0 \rightsquigarrow 1e^{i\frac{\pi}{3}}$

$k = 1 \rightsquigarrow 1e^{i\pi}$

$k = 2 \rightsquigarrow 1e^{i\frac{5\pi}{3}}$

C.S. $= \left\{0, e^{i\frac{\pi}{3}}, e^{i\pi}, e^{i\frac{5\pi}{3}}\right\}$



$A\left(\cos\left(\frac{\pi}{3}\right), \text{sen}\left(\frac{\pi}{3}\right)\right)$

$A\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$B(-1, 0)$

$C\left(\cos\left(\frac{5\pi}{3}\right), \text{sen}\left(\frac{5\pi}{3}\right)\right)$

$C\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

$$A_{[OABC]} = 2 \times A_{[OAB]} = 2 \times \frac{1 \times \frac{\sqrt{3}}{2}}{2} = \frac{\sqrt{3}}{2}\ \text{u.a.}$$

8.
$$|z+i|^2 + |z-i|^2 \leq 20$$
$$\Leftrightarrow |x + yi + i|^2 + |x + yi - i|^2 \leq 20$$
$$\Leftrightarrow |x + (y+1)i|^2 + |x + (y-1)i|^2 \leq 20$$
$$\Leftrightarrow \left(\sqrt{x^2 + (y+1)^2}\right)^2 + \left(\sqrt{x^2 + (y-1)^2}\right)^2 \leq 20$$
$$\Leftrightarrow x^2 + (y+1)^2 + x^2 + (y-1)^2 \leq 20$$
$$\Leftrightarrow 2x^2 + y^2 + 2y + 1 + y^2 - 2y + 1 \leq 20$$
$$\Leftrightarrow 2x^2 + 2y^2 \leq 18$$
$$\Leftrightarrow x^2 + y^2 \leq 9$$

Logo, o afixo de *z* pertence ao círculo de centro na origem e de raio igual a 3.

9.

9.1
$$w = \frac{z_1 \times (z_2)^2}{\overline{z}_3} = \frac{(2 - i^{19}) \times (1-i)^2}{1 + 2i} =$$
$$= \frac{(2 - i^{4\times 4+3}) \times (1 - 2i + i^2)}{1 + 2i} =$$
$$= \frac{(2 - i^3) \times (1 - 2i - 1)}{1 + 2i} =$$
$$= \frac{(2 + i) \times (-2i)}{1 + 2i} =$$
$$= \frac{-4i - 2i^2}{1 + 2i} =$$
$$= \frac{2 - 4i}{1 + 2i} =$$
$$= \frac{(2 - 4i)(1 - 2i)}{(1 + 2i)(1 - 2i)} =$$
$$= \frac{2 - 4i - 4i + 8i^2}{1^2 + (-2)^2} =$$
$$= \frac{2 - 8i - 8}{1 + 4} =$$
$$= \frac{-6 - 8i}{5} =$$
$$= -\frac{6}{5} - \frac{8}{5}i$$
$$|w| = \sqrt{\left(-\frac{6}{5}\right)^2 + \left(-\frac{8}{5}\right)^2} = \sqrt{\frac{36}{25} + \frac{64}{25}} =$$
$$= \sqrt{\frac{100}{25}} =$$
$$= \sqrt{4} =$$
$$= 2$$

Seja $\theta = \arg(w)$; $\theta \in \left]-\pi, -\frac{\pi}{2}\right[$, pois a parte real e a parte imaginária de *w* são ambas negativas.

$\text{tg}\,\theta = \dfrac{-\frac{8}{5}}{-\frac{6}{5}} = \dfrac{4}{3}$

Uma vez que a função tangente é contínua e estritamente crescente em $\left]-\pi, -\frac{\pi}{2}\right[$ e que $\text{tg}\left(-\frac{3\pi}{4}\right) = 1$, como $\frac{4}{3} > 1$, podemos garantir que $-\frac{3\pi}{4} < \theta < -\frac{\pi}{2}$.

9.2 $-z_2 \times z = -(1 - i) \times e^{i(2\theta)} =$

$= (-1 + i) \times e^{i(2\theta)} =$

$= \sqrt{2}e^{i\left(\frac{3\pi}{4}\right)} \times e^{i(2\theta)} =$

$= \sqrt{2}e^{i\left(\frac{3\pi}{4} + 2\theta\right)}$

Cálculos auxiliares

- $|-1 + i| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$
- Seja β um argumento de $-1 + i$:

$\text{tg}(\beta) = \frac{1}{-1} = -1$

$\text{Re}(w) < 0 \wedge \text{Im}(w) > 0$ β é um ângulo pertencente ao 2.º quadrante.

$\beta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$, por exemplo.

Assim, $-1 + i = \sqrt{2}e^{i\frac{3\pi}{4}}$.

$-z_2 \times z$ é um número imaginário puro se

$\frac{3\pi}{4} + 2\theta = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$.

Desta forma:

$\frac{3\pi}{4} + 2\theta = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z} \Leftrightarrow 2\theta = \frac{\pi}{2} - \frac{3\pi}{4} + k\pi,\ k \in \mathbb{Z}$

$\Leftrightarrow 2\theta = -\frac{\pi}{4} + k\pi,\ k \in \mathbb{Z}$

$\Leftrightarrow \theta = -\frac{\pi}{8} + \frac{k\pi}{2},\ k \in \mathbb{Z}$

$\theta \in \left[0, \frac{\pi}{2}\right]$, logo, para $k = 1$, tem-se que $\theta = \frac{3\pi}{8}$.

Assim, o valor de $\theta \in \left[0, \frac{\pi}{2}\right]$ para o qual $-z_2 \times z$ é um número imaginário puro é $\frac{3\pi}{8}$.

10. Opção (C)

$z = -4i = 4e^{i\left(-\frac{\pi}{2}\right)}$

$z \times e^{i\frac{\pi}{6}} = 4e^{i\left(-\frac{\pi}{2}\right)} \times e^{i\frac{\pi}{6}} =$

$= 4 \times e^{i\left(-\frac{\pi}{2} + \frac{\pi}{6}\right)} =$

$= 4 \times e^{i\left(-\frac{\pi}{3}\right)} =$

$= 4\left(\cos\left(-\frac{\pi}{3}\right) + i\,\text{sen}\left(-\frac{\pi}{3}\right)\right) =$

$= 4\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) =$

$= \frac{4}{2} - \frac{4\sqrt{3}}{2}i =$

$= 2 - 2\sqrt{3}i$

11. Comecemos por escrever $\sqrt{3} - i$ na forma trigonométrica:

$|\sqrt{3} - i| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$

Seja θ um argumento de $\sqrt{3} - i$:

$\text{tg}\,\theta = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

$\text{Re}(w) > 0 \wedge \text{Im}(w) < 0$ θ é um ângulo pertencente ao 4.º quadrante.

$\theta = -\frac{\pi}{6}$, por exemplo.

Assim, $\sqrt{3} - i = 2e^{i\left(-\frac{\pi}{6}\right)}$.

$w = \dfrac{\sqrt{3} - i}{4e^{i\left(-\frac{5\pi}{6}\right)}} = \dfrac{2e^{i\left(-\frac{\pi}{6}\right)}}{4e^{i\left(-\frac{5\pi}{6}\right)}} =$

$= \frac{1}{2}e^{i\left(-\frac{\pi}{6} + \frac{5\pi}{6}\right)} =$

$= \frac{1}{2}e^{i\frac{4\pi}{6}} =$

$= \frac{1}{2}e^{i\frac{2\pi}{3}}$

Sabemos que w é uma das raízes quintas de um certo número complexo z, logo, $z = w^5$, de onde resulta que:

$z = \left(2e^{i\frac{2\pi}{3}}\right)^5 = \left(\frac{1}{2}\right)^5 e^{i\left(5 \times \frac{2\pi}{3}\right)} =$

$= \frac{1}{32}e^{i\frac{10\pi}{3}} = \frac{1}{32}e^{i\frac{4\pi}{3}}$

$\frac{1}{32}e^{i\frac{4\pi}{3}} = \frac{1}{32}\left(\cos\left(\frac{4\pi}{3}\right) + i\,\text{sen}\left(\frac{4\pi}{3}\right)\right) =$

$= \frac{1}{32}\left(-\cos\left(\frac{\pi}{3}\right) - i\,\text{sen}\left(\frac{\pi}{3}\right)\right) =$

$= \frac{1}{32}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) =$

$= -\frac{1}{64} - \frac{\sqrt{3}}{64}i =$

Logo, $z = -\frac{1}{64} - \frac{\sqrt{3}}{64}i$.

12.

12.1 $w = \bar{z}_1 + z_1 z_2 - 6(1 + i)^2 =$

$= -2 + 3i + 4 + 7i - 6 \times 2i =$

$= 2 - 2i$

Cálculos auxiliares

- $\bar{z}_1 = -2 + 3i$
- $z_2 = \frac{3 - i}{1 + i} = \frac{(3 - i)(1 - i)}{(1 + i)(1 - i)} =$

$= \frac{3 - 3i - i + i^2}{1^2 + 1^2} = \frac{3 - 3i - i - 1}{2} =$

$= \frac{2 - 4i}{2} = 1 - 2i$

- $z_1 z_2 = (-2 + 3i) \times (1 - 2i) =$

$= -2 + 4i + 3i - 6i^2 =$

$= -2 + 7i + 6 =$

$= 4 + 7i$

- $(1 + i)^2 = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$

$|w| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

Seja θ um argumento de w:

$\text{tg}\,\theta = \frac{-2}{2} = -1$

$\text{Re}(w) > 0 \wedge \text{Im}(w) < 0$ θ é um ângulo pertencente ao 4.º quadrante.

$\theta = -\frac{\pi}{4}$, por exemplo.

Desta forma, $w = 2\sqrt{2}e^{i\left(-\frac{\pi}{4}\right)}$.

12.2 $\dfrac{i^{11}}{z_3^n} = \dfrac{-i}{\left(e^{i\left(-\frac{\pi}{5}\right)}\right)^n} = \dfrac{e^{i\frac{3\pi}{2}}}{\left(e^{i\left(-\frac{\pi}{5}\right)}\right)^n} =$

$= \dfrac{e^{i\frac{3\pi}{2}}}{e^{i\left(-\frac{n\pi}{5}\right)}} = e^{i\left(\frac{3\pi}{2} + \frac{n\pi}{5}\right)}$

Cálculos auxiliares

- $i^{11} = i^{2\times4+3} = i^3 = -i$
- $z_3 = \cos\left(\frac{\pi}{5}\right) - i\,\text{sen}\left(\frac{\pi}{5}\right) = \cos\left(\frac{\pi}{5}\right) + i\,\text{sen}\left(-\frac{\pi}{5}\right) =$
$= \cos\left(-\frac{\pi}{5}\right) - i\,\text{sen}\left(-\frac{\pi}{5}\right) =$
$= e^{i\left(-\frac{\pi}{5}\right)}$

$e^{i\left(\frac{3\pi}{2}+\frac{n\pi}{5}\right)}$ é um imaginário puro, se $\frac{3\pi}{2}+\frac{n\pi}{5} = \frac{\pi}{2}+k\pi$.
Desta forma:

$$\frac{3\pi}{2}+\frac{n\pi}{5} = \frac{\pi}{2}+k\pi,\ k\in\mathbb{Z} \Leftrightarrow \frac{n\pi}{5} = \frac{\pi}{2}-\frac{3\pi}{2}+k\pi,\ k\in\mathbb{Z}$$
$$\Leftrightarrow \frac{n\pi}{5} = -\frac{2\pi}{2}+k\pi,\ k\in\mathbb{Z}$$
$$\Leftrightarrow \frac{n}{5} = -1+k,\ k\in\mathbb{Z}$$
$$\Leftrightarrow n = -5+5k,\ k\in\mathbb{Z}$$

Atribuindo valores inteiros a k:
$k=0$: $n = -5+5\times0 = -5$ $(-5\notin\mathbb{N})$
$k=1$: $n = -5+5\times1 = 0$ $(0\in\mathbb{N})$
$k=2$: $n = -5+5\times2 = 5$
$5\in\mathbb{N}$, logo, 5 é o menor número natural positivo n de modo que $\frac{z_1+i^{11}}{z_3^n}$ seja um número imaginário puro.

Teste Final – páginas 89 a 91

1. Opção (B)

$z = a+\sqrt{3}ai,\ a>0$
$|z| = \sqrt{a^2+(\sqrt{3}a)^2} = \sqrt{a^2+3a^2} = \sqrt{4a^2} = 2|a| = 2a$
$\text{tg}\,\theta = \frac{\sqrt{3}a}{a} \wedge \theta\in1.^{\text{o}}\,\text{Q} \Leftrightarrow \text{tg}\,\theta = \sqrt{3} \wedge \theta\in1.^{\text{o}}\,\text{Q}$
$\theta = \frac{\pi}{3}$, por exemplo.
$z = 2ae^{i\frac{\pi}{3}}$
$z^3 = \left(2ae^{i\frac{\pi}{3}}\right)^3 = 2^3\times a^3e^{i\frac{3\pi}{3}} = 8a^3e^{i\pi}$

2.
$$z = \left(\frac{4e^{i\frac{\pi}{6}}+8e^{i\frac{5\pi}{6}}}{\sqrt{24}((2+i)^3-3-10i)}\right)^{12} =$$
$$= \left(\frac{4\left(\cos\left(\frac{\pi}{6}\right)+i\,\text{sen}\left(\frac{\pi}{6}\right)\right)+8\left(\cos\left(\frac{5\pi}{6}\right)+i\,\text{sen}\left(\frac{5\pi}{6}\right)\right)}{\sqrt{24}((2+i)^2(2+i)-3-10i)}\right)^{12} =$$
$$= \left(\frac{4\left(\frac{\sqrt{3}}{2}+i\times\frac{1}{2}\right)+8\left(-\frac{\sqrt{3}}{2}+i\times\frac{1}{2}\right)}{\sqrt{24}((4+4i-1)(2+i)-3-10i)}\right)^{12} =$$
$$= \left(\frac{2\sqrt{3}+2i-4\sqrt{3}+4i}{\sqrt{24}((3+4i)(2+i)-3-10i)}\right)^{12} =$$
$$= \left(\frac{-2\sqrt{3}+6i}{\sqrt{24}((6+3i+8i-4)-3-10i)}\right)^{12} =$$
$$= \left(\frac{-2\sqrt{3}+6i}{\sqrt{24}(2+11i-3-10i)}\right)^{12} =$$

Cálculo auxiliar

$|-2\sqrt{3}+6i| = \sqrt{12+36} = \sqrt{48}$
$\text{tg}\,\theta = \frac{6}{-2\sqrt{3}} \wedge \theta\in2.^{\text{o}}\,\text{Q} \Leftrightarrow \text{tg}\,\theta = \frac{-3}{\sqrt{3}} \wedge \theta\in2.^{\text{o}}\,\text{Q}$
$\Leftrightarrow \text{tg}\,\theta = -\sqrt{3} \wedge \theta\in2.^{\text{o}}\,\text{Q}$
$\theta = \frac{2\pi}{3}$, por exemplo.

$$= \left(\frac{\sqrt{48}e^{i\frac{2\pi}{3}}}{\sqrt{24}(-1+i)}\right)^{12} =$$
$$= \left(\frac{\cancel{\sqrt{2}}e^{i\frac{2\pi}{3}}}{\cancel{\sqrt{2}}e^{i\frac{3\pi}{4}}}\right)^{12} =$$
$$= \left(e^{i\left(-\frac{\pi}{12}\right)}\right)^{12} =$$
$$= e^{i(-\pi)} =$$
$$= e^{i\pi}$$
$$= -1$$

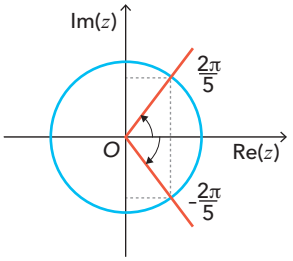
3. Opção (C)

$z = a+bi \quad a>b>0$
$$\frac{z}{i}-\bar{z} = \frac{a+bi}{i}-(a-bi) = \frac{ai^2+bi^2}{i^2}-a+bi =$$
$$= -ai+b-a+bi = \underbrace{(b-a)}_{<0}+\underbrace{(b-a)i}_{<0}$$

$b-a<0$, pois $a>b$.
Logo, o afixo de $\frac{z}{i}-\bar{z}$ pertence ao terceiro quadrante.

4. Opção (B)

$z_1 = 1e^{i0} = e^{i0}$ $\quad \overset{\frown}{AB} = \overset{\frown}{AE} = \frac{2\pi}{5}$
$z_2 = 1e^{i\frac{2\pi}{5}} = e^{i\frac{2\pi}{5}}$ $\quad \overline{OA} = 1$
$z_5 = 1e^{i\left(-\frac{2\pi}{5}\right)} = e^{i\left(-\frac{2\pi}{5}\right)}$ $\quad \overline{OB} = 1$
$z_2+z_5 = e^{i\frac{2\pi}{5}}+e^{i\left(-\frac{2\pi}{5}\right)} =$ $\quad \overline{OE} = 1$
$$= \cos\left(\frac{2\pi}{5}\right)+i\,\text{sen}\left(\frac{2\pi}{5}\right)+\cos\left(-\frac{2\pi}{5}\right)+i\,\text{sen}\left(-\frac{2\pi}{5}\right) =$$



$$= \cos\left(\frac{2\pi}{5}\right)+\cancel{i\,\text{sen}\left(\frac{2\pi}{5}\right)}+\cos\left(\frac{2\pi}{5}\right)-\cancel{i\,\text{sen}\left(\frac{2\pi}{5}\right)} =$$
$$= 2\cos\left(\frac{2\pi}{5}\right)$$

5. $\rho_1 = \sqrt{(-1)^2+\left(-\sqrt{3}\right)^2} = \sqrt{1+3} = 2$
$\text{tg}\,\theta_1 = \frac{-\sqrt{3}}{-1} \wedge \theta_1\in3.^{\text{o}}\,\text{Q} \Leftrightarrow \text{tg}\,\theta_1 = \sqrt{3} \wedge \theta_1\in3.^{\text{o}}\,\text{Q}$
$\theta_1 = \frac{4\pi}{3}$, por exemplo.
Logo, $z_1 = 2e^{i\frac{4\pi}{3}}$.
$$z_2 = \cos\left(\frac{6\pi}{5}\right)-i\,\text{sen}\left(\frac{6\pi}{5}\right) =$$
$$= \overline{\cos\left(\frac{6\pi}{5}\right)+i\,\text{sen}\left(\frac{6\pi}{5}\right)} =$$
$$= \overline{e^{i\frac{6\pi}{5}}} =$$
$$= e^{i\left(-\frac{6\pi}{5}\right)}$$

$(z_1 \times z_2)^n = \left(2e^{i\frac{4\pi}{3}} \times e^{i\left(-\frac{6\pi}{5}\right)}\right)^n =$

$= \left(2e^{i\left(\frac{4\pi}{3}-\frac{6\pi}{5}\right)}\right)^n =$

$= \left(2e^{i\frac{2\pi}{15}}\right)^n =$

$= 2^n e^{i\frac{2\pi}{15}n}$

$2^n > 0, \forall\, n \in \mathbb{N}$

$(z_1 \times z_2)^n$ é um número real positivo se $\frac{2\pi}{15}n = 2k\pi$,

$k \in \mathbb{Z} \Leftrightarrow n = 15k, k \in \mathbb{Z}$

$k = 0 \leadsto n = 0$

$k = 1 \leadsto n = 15$

O menor valor de n natural positivo é 15.

6. $\frac{3\pi}{4} - \frac{\pi}{4} = \frac{2\pi}{n} \Leftrightarrow \frac{2\pi}{4} = \frac{2\pi}{n} \Leftrightarrow n = 4$

$z_2 = 3e^{i\left(\frac{3\pi}{4}+\frac{2\pi}{4}\right)} = 3e^{i\frac{5\pi}{4}}$

$z_3 = 3e^{i\left(\frac{5\pi}{4}+\frac{2\pi}{4}\right)} = 3e^{i\frac{7\pi}{4}}$

$n = 4$ e as restantes raízes são $3e^{i\frac{5\pi}{4}}$ e $3e^{i\frac{7\pi}{4}}$.

7. $(-1 + 4i) \times i = -i + 4i^2 = -4 - i$

$(-4 - i) \times i = -4i - i^2 = 1 - 4i$

$(1 - 4i) \times i = i - 4i^2 = 4 + i$

As restantes raízes quartas de z são: $-4 - i$, $1 - 4i$ e $4 + i$.

8. $(2 + i) \times e^{i\frac{2\pi}{3}} = (2 + i) \times \left(\cos\left(\frac{2\pi}{3}\right) + i\,\text{sen}\left(\frac{2\pi}{3}\right)\right) =$

$= (2 + i) \times \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) =$

$= -1 + \sqrt{3}i - \frac{1}{2}i + \frac{\sqrt{3}}{2}i^2 =$

$= \left(-1 - \frac{\sqrt{3}}{2}\right) + \left(\sqrt{3} - \frac{1}{2}\right)i$

$(2 + i) \times e^{i\left(-\frac{2\pi}{3}\right)} = (2 + i) \times \left(\cos\left(-\frac{2\pi}{3}\right) + i\,\text{sen}\left(-\frac{2\pi}{3}\right)\right) =$

$= (2 + i) \times \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right)i =$

$= -1 - \sqrt{3}i - \frac{1}{2}i - \frac{\sqrt{3}}{2}i^2 =$

$= \left(-1 + \frac{\sqrt{3}}{2}\right) + \left(-\sqrt{3} - \frac{1}{2}\right)i$

As outras duas raízes cúbicas de z são:

$\left(-1 - \frac{\sqrt{3}}{2}\right) + \left(\sqrt{3} - \frac{1}{2}\right)i$ e $\left(-1 + \frac{\sqrt{3}}{2}\right) + \left(-\sqrt{3} - \frac{1}{2}\right)i$

9. Opção (A)

$\overline{OB}^2 = \overline{OA}^2 + \overline{AB}^2 \Leftrightarrow \overline{OB}^2 = \overline{OA}^2 + \overline{OA}^2$

$\Leftrightarrow \overline{OB}^2 = 2\,\overline{OA}^2$

$\Leftrightarrow \overline{OB} = \sqrt{2}\,\overline{OA}$

$\overline{OB} = \sqrt{2} \times \overline{OA}$

$|z_2| = \sqrt{2} \times |z_1| = \sqrt{2} \times \sqrt{(-2)^2 + 1^2} = \sqrt{2} \times \sqrt{5} = \sqrt{10}$

Sejam θ_1 e θ_2 os argumentos não negativos mínimos de z_1 e z_2, respetivamente.

$\theta_2 = \theta_1 + \frac{\pi}{4}$

$z_2 = z_1 \times \sqrt{2}e^{i\frac{\pi}{4}} = (-2 + i) \times \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\,\text{sen}\left(\frac{\pi}{4}\right)\right) =$

$= (-2 + i) \times \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) =$

$= (-2 + i) \times (1 + i) =$

$= (-2 - 2i + i + i^2) =$

$= -3 - i$

Assim, $\text{Re}(z_2) = -3$.

10. $z^5 = 81z \Leftrightarrow z^5 - 81z = 0 \Leftrightarrow z(z^4 - 81) = 0$

$\Leftrightarrow z = 0 \vee z^4 = 81$

As raízes de ordem 4 de 81 são de forma:

$\sqrt[4]{81}e^{i\left(\frac{0+2k\pi}{4}\right)}, k \in \{0, 1, 2, 3\}$

$= 3e^{i\frac{k\pi}{2}}, k \in \{0, 1, 2, 3\}$

$z_0 = 3e^{i0} = 3$

$z_1 = 3e^{i\frac{\pi}{2}} = 3i$

$z_2 = 3e^{i\pi} = -3$

$z_3 = 3e^{i\frac{3\pi}{2}} = -3i$

$\Leftrightarrow z = 0 \vee z = 3 \vee z = 3i \vee z = -3 \vee z = -3i$

C.S. $= \{0, 3, 3i, -3, -3i\}$

11. Opção (C)

$1 \leq |z - (1 + 2i)| \leq 2 \wedge -\frac{5\pi}{4} < \arg(z - (1 + 2i)) \leq \frac{\pi}{2}$

$\Leftrightarrow 1 \leq |z - 1 - 2i| \leq 2 \wedge -\frac{5\pi}{4} \leq \arg(z - 1 - 2i) \leq \frac{\pi}{2}$

12. $\text{Im}(z - 6i) \times \text{Re}(i \times \bar{z}) \leq -|-8i| \wedge \frac{\pi}{4} \leq \arg(z) \leq \frac{3\pi}{4}$

Substituindo z por $x + yi$ na condição $\text{Im}(z - 6i) \times \text{Re}(i \times \bar{z}) \leq -|-8i|$ obtemos:

$\text{Im}(z - 6i) \times \text{Re}(i \times \bar{z}) \leq -|-8i|$

$\Leftrightarrow \text{Im}(x + yi - 6i) \times \text{Re}(i \times (x - yi)) \leq -8$

$\Leftrightarrow \text{Im}(x + (y - 6)i) \times \text{Re}(xi + y) + 8 \leq 0$

$\Leftrightarrow (y - 6) \times y + 8 \leq 0$

$\Leftrightarrow y^2 - 6y + 8 \leq 0$

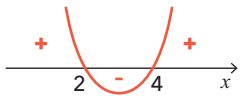
$\Leftrightarrow 2 \leq y \leq 4$

Cálculo auxiliar

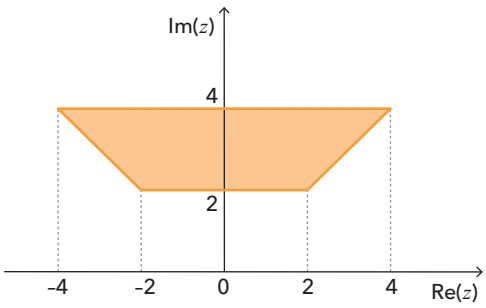
$y^2 - 6y + 8 = 0 \Leftrightarrow y = \frac{6 \pm \sqrt{36 - 4 \times 8}}{2}$

$\Leftrightarrow y = \frac{6 \pm 2}{2}$

$\Leftrightarrow y = 4 \vee y = 2$



Intersetando a região definida por esta condição com a região definida por $\frac{\pi}{4} \leq \arg(z) \leq \frac{3\pi}{4}$ obtém-se o trapézio:



$A = \frac{4 + 8}{2} \times 2 = 12$ u.a.